

PERFECT $\langle k, r \rangle$ -LATIN SQUARES.

KEVIN HARE

ABSTRACT. A **perfect $\langle k, r \rangle$ -latin square** $A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and the element $a_{i,j}$ occurs either k times in row i and r times in column j , or occurs r times in row i and k times in column j . In 1989, Cai, Kruskal, Liu and Shen studied the existence of perfect $\langle k, r \rangle$ -latin squares. Here, a simpler construction of perfect $\langle k, r \rangle$ -latin squares is given.

1. INTRODUCTION AND DEFINITIONS.

Definition 1 (Perfect $\langle k, r \rangle$ -latin square). A **perfect $\langle k, r \rangle$ -latin square** $A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and the element $a_{i,j}$ occurs either k times in row i and r times in column j , or occurs r times in row i and k times in column j .

It was shown in [1] that the above definition implies $m = \frac{n(k+r)}{2kr}$. Further, it was shown that for any k, r and h that there exists a perfect $\langle k, r \rangle$ -latin square of order $n = 2hkr(k+r)$. The goal here is to show how to construct a perfect $\langle k, r \rangle$ -latin square of order n in a simpler manner than the construction presented in [1]. First a few definitions are needed.

Definition 2 ($\langle k, r \rangle$ -pattern). A **$\langle k, r \rangle$ -pattern** $A = (a_{i,j})$ of order n with m elements is an $n \times n$ array in which each element occurs in each row and column, and if the cell (i, j) is not empty, then $a_{i,j}$ occurs either k times in row i and r times in column j , or occurs r times in row i and k times in column j .

Row and column permutations of $\langle k, r \rangle$ -patterns and perfect $\langle k, r \rangle$ -latin squares are $\langle k, r \rangle$ -patterns and perfect $\langle k, r \rangle$ -latin squares respectively. Note also that if a $\langle k, r \rangle$ -pattern has no empty cells, then it is a perfect $\langle k, r \rangle$ -latin square.

Definition 3 (Disjoint). Two $n \times n$ arrays are said to be *disjoint* if, whenever cell (i, j) of one is non-empty, then in the other array this cell is empty.

Research performed while supported by NSERC.

Observe that $H(12, 2, 0, (1, 2, 3)) + V(12, 2, 6, (3, 1, 2))$ is a $\langle 2, 1 \rangle$ -pattern.

This example shows a type V array and a type H array combine together to form a $\langle k, r \rangle$ -pattern. These $\langle k, r \rangle$ -patterns are important building blocks for perfect $\langle k, r \rangle$ -latin squares.

2. A PERFECT $\langle k, 1 \rangle$ -LATIN SQUARE.

Here, an examination of a simpler problem, that of perfect $\langle k, 1 \rangle$ -latin squares, will give an introduction to the techniques used for the problem of constructing perfect $\langle k, r \rangle$ -latin squares. The idea of this section is to fit various type H and type V arrays together to get $\langle k, 1 \rangle$ -patterns and then fit these $\langle k, 1 \rangle$ -patterns together to get a perfect $\langle k, 1 \rangle$ -latin square.

Notice that a $\langle k, 1 \rangle$ -pattern must have the property that $2k(k+1)|n$ by the comments in the previous section. So by considering $n = 2k(k+1)$ then the following is true.

Lemma 8.

$$H(2k(k+1), k, 0, (0_1, 0_2, \dots, 0_{k+1})) + \\ V(2k(k+1), k, k(k+1), (0_{k+1}, 0_1, 0_2, \dots, 0_k))$$

is a $\langle k, 1 \rangle$ -pattern.

Proof: Consider element 0_u , $1 \leq u \leq k+1$. Element 0_u occurs in cells $((k+1)t+u, s+(k+1)t+u)$ of the type H array and cells $(s+(k+1)t+u+1, k(k+1)+k+(k+1)t+u)$ of the type V array, where $0 \leq s \leq k-1$ and $0 \leq t \leq 2k-1$. Next it is shown that this sum satisfies the conditions of a $\langle k, 1 \rangle$ -pattern for the element 0_u .

0_u will occur in every row, as it occurs in rows $s+(k+1)t+u+1$, and also in rows $(k+1)t+u$, where $0 \leq s \leq k-1$ and $0 \leq t \leq 2k-1$. Similarly, 0_u will occur in every column.

Consider an element 0_u from the type H array in cell $((k+1)t+u, s+(k+1)t+u)$. Then in this row, there will be k occurrences of 0_u ; and in this column 1 occurrence of 0_u . The type V array cannot contribute to this row as $(k+1)t+u \not\equiv s+(k+1)t'+u+1 \pmod{k+1}$, for $0 \leq s \leq k-1$ and $0 \leq t, t' \leq 2k-1$. Further the type V array cannot contribute to this column as $k(k+1)+k+(k+1)t+u \not\equiv s+(k+1)t'+u \pmod{k+1}$ for $0 \leq s \leq k-1$ and $0 \leq t, t' \leq 2k-1$.

A similar argument can be made for the 0_u that are elements of a type V array.

Hence this is a $\langle k, 1 \rangle$ -pattern. ■

Corollary 9.

$$H(2k(k+1), k, p, (0_1, 0_2, \dots, 0_{k+1})) + \\ V(2k(k+1), k, p+k(k+1), (0_{k+1}, 0_1, 0_2, \dots, 0_k))$$

is a $\langle k, 1 \rangle$ -pattern.

Here it is worth observing that the non-empty cells of a type V array coincide with those of a type H array if and only if the arrays have the same n , k and p values. It follows that whether two arrays are disjoint depends only on the n , k and p values and not on the types of the arrays.

Lemma 10. *The array*

$$(1) \quad H(2kr(k+r), k, ik, (i_1, \dots, i_{k+r}))$$

is disjoint from

$$(2) \quad H(2kr(k+r), k, jk, (j_1, \dots, j_{k+r}))$$

for $i \neq j$ and $0 \leq i, j \leq k+r-1$.

Proof: This follows by noting that there are elements in cells $((k+r)t+u, ik+s+(k+r)t+u)$ for $0 \leq t \leq 2kr-1$, $0 \leq s \leq k-1$ in array 1 and in cells $((k+r)t+u, jk+s+(k+r)t+u)$ for $0 \leq t \leq 2kr-1$, $0 \leq s \leq k-1$ in array 2. So given a particular row, $(k+r)t+u$, t and u become fixed. As $i \neq j$, the columns of non-empty cells in this row are distinct. Hence these two arrays are disjoint. ■

In the following theorem and the subsequent results it is convenient to increment i_1 etc. along with i so that, in the following case, the resulting array is defined on the symbols $\{0_1, \dots, 0_{k+1}, 1_1, \dots, 1_{k+1}, \dots, k_1, \dots, k_{k+1}\}$.

Theorem 11.

$$\sum_{i=0}^k (H(2k(k+1), k, ik, (i_1, i_2, \dots, i_{k+1})) + \\ V(2k(k+1), k, k(k+1) + ik, (i_{k+1}, i_1, i_2, \dots, i_k)))$$

is a perfect $\langle k, 1 \rangle$ -latin square.

Proof: Taking $r = 1$ in Lemma 10, ensures that this sum is well defined. The number of non-empty cells in a type V, or type H array is $kn = 2k^2(k+1)$. As there are $k+1$ type V arrays, and $k+1$ type H arrays, there are $4k^2(k+1)^2$ cells covered in this sum (which is the total number of cells in the array). Lemma 8 gives that this is a $\langle k, 1 \rangle$ -pattern and since there are no empty cells it is a perfect $\langle k, 1 \rangle$ -latin square. ■

3. A PERFECT $\langle k, r \rangle$ -LATIN SQUARE.

This section proceeds much the same as Section 2, but investigates the more general case of $\langle k, r \rangle$ -patterns.

It can easily be seen that if k and r are not relatively prime, then the problem can be solved for k' and r' , ($k' = \frac{k}{\gcd(k,r)}$, $r' = \frac{r}{\gcd(k,r)}$) and expanded to solve the problem for k and r by Kronecker products. (See Corollary 16 for a more detailed description of this.)

Lemma 12. *Assume k and r are relatively prime. Choose j_l such that $j_l \equiv -l \pmod{k+r}$, $j_l \equiv 0 \pmod{k}$, and $j_l \equiv l \pmod{r}$ for $0 \leq l \leq r-1$. Set $j'_l = j_l + rk(k+r)$. Then*

$$\sum_{l=0}^{r-1} (H(2kr(k+r), k, j_l, (0_{k+r-l+1}, 0_{k+r-l+2}, \dots, 0_{k+r}, 0_1, 0_2, \dots, 0_{k+r-l})) \\ + V(2kr(k+r), k, j'_l, (0_{k+1}, 0_{k+2}, \dots, 0_{k+r}, 0_1, 0_2, \dots, 0_k)))$$

is a $\langle k, r \rangle$ -pattern.

Proof: First, observe that such j_l can be chosen from the Chinese Remainder Theorem, as r , k and $r+k$ are relatively prime. Also Lemma 10 guarantees that this sum is well defined.

Consider an element 0_u . It lies in cell: $((k+r)t+u+l, j_l+s+(k+r)t+u+l)$ from the type H array, and $(s+(k+r)t+u+r, j'_l+k+(k+r)t+u+r-1)$ from the type V array, $0 \leq s \leq k-1$, $0 \leq t \leq 2kr-1$ and $0 \leq l \leq r-1$.

Consider row $(k+r)t+u+l$. This contains k occurrences of 0_u , namely in cells, $((k+r)t+u+l, j_l+s+(k+r)t+u+l)$, from the type H array. No type V array can contribute a 0_u to this row as $s'+(k+r)t'+u+r \not\equiv (k+r)t+u+l \pmod{2kr(k+r)}$, for $0 \leq s' \leq k-1$ and $0 \leq t, t' \leq 2kr-1$. A type H array with a different l value cannot contribute to this row as $(k+r)t+u+l \not\equiv (k+r)t'+u+l' \pmod{k+r}$, for $0 \leq t, t' \leq 2kr-1$.

Consider column $j_{l_0}+s+(k+r)t+u+l_0$. There are r occurrences of 0_u here, namely, in cell $((k+r)t_l+u+l, j_l+s+(k+r)t_l+u+l)$ from the type H array, for each l , and appropriate values of t_l . (Notice that $j_l+l \equiv 0 \pmod{k+r}$, hence t'_l can be chosen such that $j_l+s+(k+r)t_l+u+l = j_{l_0}+s+(k+r)t+u+l_0$.) The type V array does not contribute to this sum because $j_l+s+(k+r)t+u+l \not\equiv j'_{l'}+k+(k+r)t'+u-1'+r \pmod{k+r}$.

Similarly, checking rows $s+(k+r)t+u+r$ and columns $j'_l+k+(k+r)t+u-1$ gives that 0_u occurs r and k times respectively.

Lastly, 0_u will occur in every row. To see this, consider a row a . If $a-u \equiv 0, 1, \dots, r-1 \pmod{k+r}$, then it will be found in a type H array of row value $(k+r)t+u+l$. If $a-u \equiv r, r+1, \dots, r+k-1 \pmod{k+r}$, then it will be found in a type V array of row value $s+(k+r)t+u+r$ with $s \equiv a-u-r \pmod{k+r}$. Similarly, it can be shown that 0_u will occur in every column.

Hence this is a $\langle k, r \rangle$ -pattern. ■

Corollary 13. *Assume that k and r are relatively prime. Select j_l as before. Then*

$$\sum_{l=0}^{r-1} (H(2kr(k+r), k, p + j_l, (0_{k+r-l+1}, 0_{k+r-l+2}, \dots, 0_{k+r}, 0_1, 0_2, \dots, 0_{k-l})) \\ + V(2kr(k+r), k, p + j'_l, (0_{k+1}, 0_{k+2}, \dots, 0_{k+r}, 0_1, 0_2, \dots, 0_k)))$$

is a $\langle k, r \rangle$ -pattern.

The following theorem is equivalent to Theorem 3.1 of [1].

Theorem 14. *Assume k and r are relatively prime. Pick j_l, j'_l as above. Then:*

$$\sum_{i=0}^{k+r-1} \sum_{l=0}^{r-1} (H(2kr(k+r), k, ikr + j_l, (i_{k+r-l+1}, i_{k+r-l+2}, \dots, i_{k+r}, \\ i_1, i_2, \dots, i_{k+r-l})) + V(2kr(k+r), k, ikr + j'_l, (i_{k+1}, i_{k+2}, \dots, i_{k+r}, \\ i_1, i_2, \dots, i_k)))$$

is a perfect $\langle k, r \rangle$ -latin square.

Proof: From Lemma 10 this sum is well defined.

There are $r(k+r)$ type H arrays, and $r(k+r)$ type V arrays. There are $2kr(k+r)k$ cells covered by each of the type H and type V arrays. Hence there are $4k^2r^2(k+r)^2$ cells covered in total, which is equal to the total number of cells in the $2kr(k+r) \times 2kr(k+r)$ array. Lemma 12 gives that this is a $\langle k, r \rangle$ -pattern and since there are no empty cells it is a perfect $\langle k, r \rangle$ -latin square. ■

Example 15. *Consider $k = 3, r = 2$. Then $j_0 = 0, j_1 = 4, j'_0 = 10$ and $j'_1 = 14$. So taking i_u to be the $i(k+r) + u$ letter of the alphabet, the perfect*

where the $b_{i,j}$ are all equal to the same constant, gives a perfect $\langle k, r \rangle$ -latin square of order $2k'r'(k'+r')$, call this C . By taking the Kronecker product of C with any latin-square of order h , a perfect $\langle k, r \rangle$ -latin square of order N results. ■

4. ACKNOWLEDGMENTS

I would like to thank Dr. Kathy Heinrich for all of her support with respect to this paper.

REFERENCES

- [1] Y. Z. Cai, Clyde P. Kruskal, C. L. Liu, and Xiaojun Shen. Generalized latin squares I. *Discrete Applied Mathematics*, 25:155–178, 1989.

CENTRE FOR EXPERIMENTAL AND CONSTRUCTIVE MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA, KGHARE@CECM.SFU.CA