INFINITE BARKER SERIES

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Abstract. We say a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is a *Littlewood polynomial* if $a_k = \pm 1$ for $0 \leq k \leq n$. Let $p(z)p(1/z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_{-n} z^{-n}$. It is easy to show that $c_0 = n+1$. We say that $p(z)$ is a *Barker polynomial* if $|c_k| \leq 1$ for $k \neq 0$. There are only 8 known Barker polynomials (normalized to have $a_n = a_{n-1} = 1$). There are many results known about the existence and non-existence of Barker polynomials for various degrees. This paper deals with the infinite case, when $f(z) = \pm 1 \pm z \pm z^2 \pm \cdots$ is a power series with ± 1 coefficients. We give a complete description of all Barker series.

1. Introduction and Preliminary results

We say a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is a *Littlewood polynomial* if $a_k = \pm 1$ for $0 \leq k \leq n$. Let $p(z)p(1/z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_{-n} z^{-n}$. These c_k are known as the *aperiodic autocorrelation coefficients* (sometimes called the acyclic autocorrelation coefficients). It is easy to show that $c_0 = n+1$. We say that $p(z)$ is a *Barker polynomial* if $|c_k| \leq 1$ for $k \neq 0$. It is clear that if $p(z)$ is a Barker polynomial, then so is $\pm p(\pm z)$. There are only 8 known Barker polynomials (normalized to have $a_n = a_{n-1} = 1$). They are

$$\begin{aligned}
&z + 1,\\
&z^2 + z - 1,\\
&z^3 + z^2 - z + 1,\\
&z^3 + z^2 + z - 1,\\
&z^4 + z^3 + z^2 - z + 1,\\
&z^6 + z^5 + z^4 - z^3 - z^2 + z - 1,\\
&z^{10} + z^9 + z^8 - z^7 - z^6 - z^5 + z^4 - z^3 - z^2 + z - 1,\\
&z^{12} + z^{11} + z^{10} + z^9 + z^8 - z^7 - z^6 + z^5 + z^4 - z^3 + z^2 - z + 1.
\end{aligned}$$

There are many results known about the existence and non-existence of Barker polynomials for various degrees. It is straightforward to show that if $p(z)$ is of degree n then $c_0 = n+1$, $c_k = 0$ when $k \not\equiv n \bmod 2$ and $c_k = \pm 1$ when $k \equiv n \bmod 2$. For n even, we can say exactly when $c_k = 1$ and when $c_k = -1$. As an example of a more sophisticated result, it is know that if a Barker polynomial of degree n exists, other than those listed above, then $n+1 = 4N^2$ where N is odd and $N \geq 55$ [1, 5, 6, 7]. This was improved in [2, 4] to show that either $N \geq 5000$ or N is one of 6 exceptional values, the smallest of which is 689.

In this paper we look at the infinite variation of this problem. We call $f(z) = \pm 1 \pm z \pm z^2 \pm \cdots$ a *Littlewood series.* We denote the set of all such Littlewood series by $\mathcal{L}$. Clearly if $f(z) = \pm 1 \pm z \pm z^2 \pm \cdots$ is a Littlewood series, then it

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does not make sense to talk about $f(z)f(1/z)$, as this is analytic nowhere. We first recall that all even degree Barker polynomials are skew symmetric, that is $z^n p(1/z) = (-1)^{n/2} p(-z)$, [7]. Hence, we will initially define a *skew-symmetric Barker series* as a Littlewood series $f(z) = \pm 1 \pm z \pm z^2 \pm \cdots$ such that $f(z)f(-z) = c_0 + c_1 z + c_2 z^2 + \cdots$, and where $|c_k| \leq 1$ for all k. For convenience we will typically omit the "skew-symmetric" and just call such an $f(z)$, a *Barker series*. Clearly $f(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$ is in $\mathcal{P}$. This example is called the *trivial Barker series*. Further, if $f(z) \in \mathcal{P}$ then $\pm f(\pm z) \in P$, hence $f(z)$ can be normalized such that $a_0 = a_1 = 1$.

There are a number of easy results that allow us to improve this definition.

Lemma 1.1. *Let* $P(z) = \sum c_i z^i$, *and* $f(z) = \sum a_i z^i$ *be power series with integer coefficients such that* $f(z)f(-z) = P(z)$. *Then* $c_{2k+1} = 0$ *and* $a_k \equiv c_{2k} \bmod 2$.

Proof. By noticing that $f(x)f(-x)$ is an even function, we see that $c_{2k+1} = 0$ for all k. Further, noticing that

$$c_{2k} = \sum_{i=0}^{2k} a_{2k-i} a_i (-1)^i = 2\left(\sum_{i=0}^{k-1} a_{2k-i} a_i (-1)^i\right) + a_k^2 (-1)^k \equiv a_k^2 \equiv a_k \pmod 2$$

gives the desired result. □

It is worth noting that if $a_k = \pm 1$, and $|c_k| \leq 1$, as is the case for Barker series, then this shows that $c_{2k+1} = 0$ and $c_{2k} = \pm 1$.

Now, knowing what a_k is equivalent to modulo 2 actually gives us information on what the c_k must be equivalent to modulo 4.

Lemma 1.2. *Let* $P(z) = \sum c_{2i} z^{2i}$ *and* $f(z) = \sum a_i z^i$ *be power series with integer coefficients satisfy* $f(z)f(-z) \equiv P(z) \bmod 4$. *Then for all* $g(z) \equiv f(z) \bmod 2$ *we have* $g(z)g(-z) \equiv P(z) \bmod 4$.

Proof. We see that

$$c_{2k} = \sum_{i=0}^{2k} a_i a_{2k-i} (-1)^i = a_k^2 + \sum_{i=0}^{k-1} 2 a_i a_{2k-i} (-1)^i$$

By noticing that the value of a_k modulo 2 uniquely determines the value of $a_k^2 \bmod 4$, and that the value of $a_i a_{2k-1} (-1)^i \bmod 2$ uniquely determines the value of $2 a_i a_{2k-1} (-1)^i \bmod 4$, we get the desired result. □

Consider again the initial definition of a Barker series, that $f(z)f(-z) = c_0 + c_1 z + \cdots$ with $|c_k| \leq 1$. By the previous remark we get that $c_{2k} = \pm 1$ and $c_{2k+1} = 0$. By considering the trivial Barker series $f(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$ where $f(z)f(-z) = \frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots$, we get that $c_{2k} = 1$ for all Barker series, as $c_{2k} \equiv 1 \bmod 4$ and $c_{2k} = \pm 1$.

As a result of Lemmas 1.1 and 1.2 we can now modify our our original definition to get:

Definition 1. *We say* $f(z)$ *is a* (skew-symmetric) Barker series *if* $f(z) \in \mathcal{L}$ *and*

$$f(z)f(-z) = 1 + z^2 + z^4 + \cdots = \frac{1}{1-z^2}. \tag{1}$$

We denote the set of all such Barker series by $\mathcal{P}$.

The main result of this paper is to give a complete description all Barker series.

Theorem 1.3. *Let* $f(z) = \sum a_i z^i$ *be a non-trivial Barker series. Further, assume* $a_0 = a_1 = 1$. *Then there exists a unique odd number* $r \geq 3$ *such that* $f(z) = (1 + z + \cdots + z^{r-1})f_2(-z^r)$ *where* $f_2(z) = a'_0 + a'_1 z + \cdots$ *is a (possibly trivial) Barker series with* $a'_0 = a'_1 = 1$.

This is proved in Section 1.3. This result allows us to uniquely associate every normalized $f(z) \in \mathcal{P}$ with a (possibly infinite) sequence of odd numbers greater than or equal to 3. For convenience we denote the trivial Barker series by

$$f_{\{\}}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}.$$

A few examples of these decompositions include

$$\begin{aligned}
f_{\{3\}} &= (1 + z + z^2)f_{\{\}}(-z^3) \\
&= (1 + z + z^2)(1 - z^3 + z^6 - z^9 + z^{12} - z^{15} + z^{18} - \cdots) \\
&= 1 + z + z^2 - z^3 - z^4 - z^5 + z^6 + z^7 + z^8 - z^9 - z^{10} - z^{11} + \cdots \\
f_{\{5,3\}} &= (1 + z + z^2 + z^3 + z^4)f_{\{3\}}(-z^5) \\
&= (1 + z + z^2 + z^3 + z^4)(1 - z^5 + z^{10})f_{\{\}}(z^{15}) \\
&= (1 + z + z^2 + z^3 + z^4)(1 - z^5 + z^{10})(1 + z^{15} + z^{30} + \cdots) \\
&= (1 + z + z^2 + z^3 + z^4)(1 - z^5 + z^{10} + z^{15} - z^{20} + z^{25} + z^{30} + \cdots) \\
&= 1 + z + z^2 + z^3 + z^4 - z^5 - z^6 - z^7 - z^8 - z^9 + z^{10} + z^{11} + \cdots \\
f_{\{3,3,3,\cdots\}} &= \prod_{i=0}^{\infty}(1 + (-1)^i z^{3^i} + z^{2\cdot 3^i})
\end{aligned}$$

We see that for a finite sequence, the Barker series is associated to a product of rational function, hence it is rational. In the case where f is associated to an infinite sequence of odd numbers, then the resulting series is not rational. By a results of [3], this means that the series is transcendental.

The idea of Barker series can be generalized if we allow more flexibility to the coefficients of $f(z)$ or alter the series $P(z)$ where $f(z)f(-z) = P(z)$. We call $f(z) = \sum a_i z^i$ with $a_i \in \{0, \pm 1\}$ a *height 1 Taylor series*. We denote the set of all such height 1 Taylor series by $\mathcal{H}$. We say that $f(z)$ is a *generalized skew-symmetric Barker series associated to* $P(z)$ if $f(z) \in \mathcal{H}$ and $f(z)f(-z) = P(z)$. As before, we typically omit the term "skew-symmetric". We denote the set of all such generalized Barker series associated to $P(z)$ as $\mathcal{P}_P$.

An obvious question is, what if we wanted to find $f(z) \in \mathcal{H}$ such that $f(z)f(-z) = P(z)$? If $P(z)$ is finite, then from Lemma 1.1 only a finite number of terms of $f(z) \in \mathcal{H}$ are non-zero. In this case $f(z)$ will be a height one polynomial, and there will be only a finite number of solutions, (all of which are easily enumerated by brute force searching).

The next useful lemma is true in much greater generality than we give here, and comes from noticing that if ζ is a pole or zero of $f(z)$ then so is $\overline{\zeta}$. Further $-\zeta$ and $-\overline{\zeta}$ are a pole or zero of $f(-z)$ with the same order. This will allow us to say specific things about generalized Barker series associated to specific $P(z)$.

Lemma 1.4. *Let* $f(z) \in \mathcal{H}$. *Further let* $f(z)f(-z) = P(z)$. *Then all zeros and poles of* $P(z)$ *on the imaginary axis have even order. If* γ *is a pole or zero off of the imaginary axis, then* $-\gamma$, $\overline{\gamma}$ *and* $-\overline{\gamma}$ *are also poles or zeros of the same order.*

Consider the case when $P(z)$ is periodic. This gives us that $P(z) = \frac{p(z)}{1-z^m}$ for some $p(z)$. We see from Lemma 1.4 that either $(1 - z^m) | p(z)$ or $m = 2n$ is even, and further n is odd.

We will discuss three non-trivial cases in Section 3 of this paper. The first is $P(z) = \frac{1}{1-z^{2n}}$. This is the simplest periodic case that can be examined. The result is not surprising in this case, and is given for completeness. The second case is, in some sense, the second simplest periodic case, that of $P(z) = \frac{1-z^{2k}}{1-z^{2n}}$. The last case is not periodic. In some sense it is "close" to a Barker sequence, in that only one coefficient is "wrong". This is the case $P(z) = 1-3z^2+z^4+z^6+z^8+\cdots = \frac{1}{1-z^2}-4z^2$. What is surprising is that there are exactly 4 solutions in this case. These results are summarized by

Theorem 1.5. *Let $\mathcal{P}_P$ be defined as above. Then*

$$\mathcal{P}_{\frac{1}{1-z^{2n}}} = \{f(z) \mid f(z) = g(z^n), g(z) \in \mathcal{P}\} \tag{2}$$

$$\mathcal{P}_{\frac{1-z^{2k}}{1-z^{2n}}} = \{f(z) \mid f(z) = \pm(1 \pm z^k) g(\pm z^n), g(z) \in \mathcal{P}\} \tag{3}$$

$$\mathcal{P}_{\frac{1}{1-z^2}-4z^2} = \{\pm f_0(\pm z) \mid f_0(z) = 1 + z - z^2 - z^3 - z^4 - z^5 - \cdots\} \tag{4}$$

The problem in equation (4) of when $P(x) = 1 - 3x^2 + x^4 + x^6 + x^8 + \cdots$ can be looked at more generally. This was a question communicated to the author by Peter Borwein. The original question was, what if $P(z) = \frac{1}{1-z} - 4z^k$? This is really as close to a Barker series as you can get, in some sense, yet still not be Barker series. One term is -3, instead of 1. It appears that $k = 2$ is the only value of k for which there is a solution, but the author knows of no way of proving this. Some results can be still shown. If k is odd, then there is not a solution by Lemma 1.1 as $P(z)$ needs to be even. If $k \equiv 0 \bmod 4$ then there are the wrong number of roots on the imaginary axis, hence there are no solutions by Lemma 1.4. If $k = 6, 10, 14, 18, 22$ or 26 then we can computationally verify that there are no solutions. For example for $k = 6$, we computationally show that all Littlewood series $f(z)$ truncated to degree 77, cannot satisfy the equations $f(z)f(-z) = P(z) \bmod z^{77}$ using a tree pruning search of all potential Littlewood series.. For $k = 26$, we need to go up to degree 257, and after that memory constraints did not allow further verification.

2. Proof of Theorem 1.3.

In this section, we give a complete description of Barker series $f(z)$. We will use a method that takes advantage of the series structure, which will be extended to other cases in the following section.

If $f(z)$ is a Barker series, then so is $\pm f(\pm z)$. So we can assume with loss of generality that $a_0 = a_1 = 1$.

Rewriting, we get

$$\begin{array}{rrcl} & f(z)f(-z) & = & \frac{1}{1-z^2} \\ \Longrightarrow & (f(z)(1-z))\,(f(-z)(1+z)) & = & 1 \\ \Longrightarrow & g(z)g(-z) & = & 1 \end{array}$$

where $g(z) = f(z)(1-z) = \sum b_i z^i$.

Now, because we are assuming that $a_0 = a_1 = 1$, we get that $b_0 = 1$ and $b_1 = 0$. Further, we get that $b_k \in \{-2, 0, 2\}$ for all $k \geq 2$. We also see that if b_i and b_j are non-zero terms, such that all $b_k = 0$ for all k, $i < k < j$, then $b_i = -b_j$. This occurs if $a_{i-1} = a_i = \cdots = a_{j-1} = -a_j$. If $b_k = 0$ for all $k \geq 2$, (i.e. $g(z) = 1$) then we have $f(x) = \frac{1}{1-x}$ and we are done. So, assume for now that there exists an $r > 1$ such that $b_r \neq 0$. Our first goal will be to show that r is odd.

The next result follow from expanding $g(z)g(-z)$ and matching up coefficients. This result is true for general integer Taylor series.

Lemma 2.1. *Let* $g(z) = \sum b_i z^i$ *be an integer Taylor series such that* $g(z)g(-z) = 1$. *Then*

$$b_0 = \pm 1$$

and for all $k \geq 1$ *we have*

$$2b_{2k} = -\sum_{i=1}^{2k-1} b_i b_{2k-i}(-1)^i = -\left(2\sum_{i=1}^{k-1} b_i b_{2k-i}(-1)^i\right) - b_k^2(-1)^k \tag{5}$$

It is worth observing that the set of such polynomials is non-empty. A simple example of such a function is $g(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots$. It is also worth noting that Lemma 2.1 is in fact if and only if.

Corollary 2.2. *Let* $g(z) = \sum b_i z^i = \pm 1 + b_r z^r + b_{r+1} z^{r+1} + \cdots$ *be an integer polynomial such that* $g(z)g(-z) = 1$, *and* $b_r \neq 0$. *Then* r *is odd.*

Proof. To prove that r is odd, assume instead that $r = 2k$ is even. But then by equation (5) we get that $2b_{2k} = 0$, a contradiction. So r is odd. ☐

The next result shows that $g(z)$ is in fact a series in z^r.

Lemma 2.3. *Let* $g(z) = f(z)(1-z) = 1 + b_r z^r + b_{r+1} z^{r+1} + \cdots$ *be defined as above. Let* $r > 1$ *be minimal, and odd, such that* $b_r \neq 0$. *Then for all* k *such that* $b_k \neq 0$ *we have* $r|k$.

Proof. We will proceed by induction.

Assume first that $k = 2m$ is even and $r \nmid 2m$. Further, assume that for all $\ell < 2k$ that $b_\ell \neq 0$ implies $r|\ell$. This implies that $b_i b_{2m-i} = 0$ for all $1 \leq i \leq 2m-1$. Then by Lemma 2.1 we have that:

$$\begin{aligned} 2b_{2k} &= -\sum_{i=1}^{2k-1} b_i b_{2k-i}(-1)^i \\ &= 0 \end{aligned}$$

as required.

Now assume that k is odd, and that $r \nmid k$. Further, assume that for all $\ell < k$ that $b_\ell \neq 0$ implies $r|\ell$. We will now use equation (5) on $k+r$. Notice for $i < r$ and for $i > k$ we get $b_i b_{k+r-i} = 0$ (as r is the minimal value greater than 0 for which $b_r \neq 0$). Secondly, for $r < i < k$ we have that $b_i b_{k+r-i} = 0$ as one of the two of i or $k + r - i$ is not divisible by r. So we get

$$2b_{k+r} = -\sum_{i=1}^{k-1} b_i b_{k-i}(-1)^i = 2b_r b_k (-1)^r$$

But we see that the left hand side takes the values 0 or ± 4. The right hand side takes the values 0 or ± 8. This implies that $b_{k+r} = b_k = 0$ as required. ☐

This then tells us that $g(z)$ looks like

$$g(z) = 1 + b_r z^r + b_{2r} z^{2r} + b_{3r} z^{3r} + \cdots .$$

As the $b_{k\cdot r}$ are 0 or ± 2, with $b_r = -2$, and consecutive non-zero terms alternate in sign, we see that this can be rewriting as

$$\begin{aligned} g(z) &= (1 - z^r) - (z^r - z^{2r}) \pm (z^{2r} - z^{3r}) \cdots \\ &= (1 - z^r)(1 - z^r \pm z^{2r} \pm z^{3r} \pm z^{4r} \cdots \end{aligned}$$

Dividing by $1 - z$ gives the result that $f(z) = (1 + z + \cdots + z^{r-1}) f_2(-z^r)$ for some $f_2(z) \in \mathcal{L}$. It remains to show that $f_2(z) \in \mathcal{P}$.

We see that $f(z) = (1 + z + \cdots + z^{r-1}) f_2(-z^r)$ from the previous comment. To notice that $f_2(z)$ is a Barker series, we notice first that

$$\begin{aligned} f(z)f(-z) &= 1 + z^2 + z^4 + z^6 + \cdots \\ &= (1 + z^2 + \cdots + z^{2(r-1)})(1 + z^{2r} + z^{4r} + \cdots) \end{aligned}$$

and further that

$$\begin{aligned} f(z)f(-z) &= (1 + z + \cdots + z^{r-1})(1 - z + \cdots + z^{r-1}) f_2(z^r) f_2(-z^r) \\ &= (1 + z^2 + z^4 + \cdots + z^{2(r-1)}) f_2(z^r) f_2(-z^r) \end{aligned}$$

from which we conclude that

$$f_2(z^r) f_2(-z^r) = 1 + z^{2r} + z^{4r} + \cdots$$

as required.

3. Proof of Theorem 1.5

The structure of $\mathcal{P}_P$ where $P = \frac{1}{1-z^{2n}}$.

Let $f(z) = \sum a_i z^i \in \mathcal{H}$. Let $P = \frac{1}{1-z^{2n}}$ By Lemma 1.1 we see that $a_i = 0$ for $i \not\equiv 0 \bmod r$ Hence we can write $f(z) = g(z^n)$ for some $g(z) \in \mathcal{H}$. Hence we have $f(z) = g(z^n)$, such that $g(z)g(-z) = \frac{1}{1-z^2}$. This implies that

$$\mathcal{P}_{\frac{1}{1-z^{2n}}} = \{f(z) \mid f(z) = g(z^n), g(z) \in \mathcal{P}\}$$

The structure of $\mathcal{P}_P$ where $P = \frac{1-z^{2k}}{1-z^{2n}}$.

There are a few things we can notice before we begin. First, we can assume with loss of generality that $\gcd(k, n) = 1$, else we solve the case for $\frac{1-z^{2k'}}{1-z^{2n'}}$ and evaluate those solutions at $z^{\gcd(k,n)}$. Next, we can assume that both k and n are odd numbers, otherwise we have the wrong number of zeros or poles at $\pm i$ by Lemma 1.4.

We see from Lemma 1.1 that $f(z)$ will look like $f(z) = g_1(z^n) + z^k g_2(z^n)$ for some $g_1(z)$ and $g_2(z)$ in $\mathcal{L}$. By noticing that

$$f(z)f(-z) = g_1(z^n)g_1(-z^n) + z^k(g_1(-z^n)g_2(z^n) - g_1(z^n)g_2(-z^n)) - z^{2k} g_2(z^n)g_2(-z^n)$$

By matching up coefficients, we notice that

$$g_1(z^n)g_1(-z^n) = 1 + z^{2n} + z^{4n} + z^{6n} + \cdots$$

and

$$g_2(z^n)g_2(-z^n) = 1 + z^{2n} + z^{4n} + z^{6n} + \cdots$$

This implies that g_1 and g_2 are in $\mathcal{P}$. Further we notice that

$$g_1(-z^n)g_2(z^n) = g_1(z^n)g_2(-z^n),$$

which implies that

$$g_1(-z)g_2(z) = g_1(z)g_2(-z). \tag{6}$$

It is worth observing that certain combinations of initial coefficients are not permitted. For example if $g_1(z) = 1 + z$ then $g_2(z) \neq 1 - z$, and $g_2(z) \neq -1 + z$. This allows us to simultaneously normalize $g_1(z)$ and $g_2(z)$. Let $\widetilde{g_1}(z)$ be normalized such that $\widetilde{g_1}(x) = 1 + z \pm z^2 \pm \cdots$ and similarly $\widetilde{g_2}(z)$. We see that

$$\widetilde{g_1}(-z)\widetilde{g_2}(z) = \widetilde{g_1}(z)\widetilde{g_2}(-z).$$

As $\widetilde{g_1}(z) \in \mathcal{P}$, we see from Theorem 1.3 that either $\widetilde{g_1}(z)$ is trivial, or there exists an r_1 such that $\widetilde{g_1}(z) = (1 + z + \cdots + z^{r_1-1})\widetilde{g_1}'(-z^{r_1})$ with $\widetilde{g_1}'(z) \in \mathcal{P}$. Assume first that $\widetilde{g_1}(z) = \frac{1}{1-z}$ is trivial. By cross multiplying, we get

$$\widetilde{g_2}(z)(1 - z) = \widetilde{g_2}(-z)(1 + z)$$

Let $\widetilde{g_2}(z) = 1 + z + a_2z^2 + a_3z^3 + \cdots$, so this gives us

$$1+(a_2-1)z^2+(a_3-a_2)z^3+(a_4-a_3)z^4+\cdots = 1+(a_2-1)z^2-(a_3-a_2)z^3+(a_4-a_3)z^4-\cdots$$

Matching up coefficients, we get that $a_{2m+1} = a_{2m}$ for all m. Further we have that $a_1 = 1$. Now, we see from Theorem 1.3 that if any $a_k = -1$ with k minimal, then k would be odd. But this can't happen as $a_k = a_{k-1}$ for k odd. Hence we have that $\widetilde{g_2}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$ is trivial, (hence $\widetilde{g_1}(z) = \widetilde{g_2}(z)$). Similarly if $\widetilde{g_2}(z)$ is trivial, then $\widetilde{g_1}(z)$ is trivial.

Now assume that there instead exists some r_1 and r_2 such that $\widetilde{g_1}(z) = (1 + z + z^2 + \cdots + z^{r_1-1})\widetilde{g_1}'(-z^{r_1})$ and $\widetilde{g_2}(z) = (1 + z + z^2 + \cdots + z^{r_2-1})\widetilde{g_2}'(-z^{r_2})$. It is easy to see that $r_1 = r_2$, and that this reduces to

$$\widetilde{g_1}'(-z)\widetilde{g_2}'(z) = \widetilde{g_1}'(z)\widetilde{g_2}'(-z)$$

Hence by induction we have that $\widetilde{g_1}(z) = \widetilde{g_2}(z)$.

Hence

$$\mathcal{P}_{\frac{1-z^{2k}}{1-z^{2n}}} = \{f(z) \mid f(z) = \pm(1 \pm z^k)g(\pm z^n), g(z) \in \mathcal{P}\}$$

The structure of $\mathcal{P}_P$ where $P = 1 - 3z^2 + z^4 + z^6 + z^8 + \cdots = \frac{1}{1-z^2} - 4z^2$.

Here we have that $P = 1 - 3z^2 + z^4 + z^6 + z^8 + \cdots$. So this is "close" to a Barker series, with one term being -3 instead of 1. As before, we normalize $f(z)$ such that $f(z) = 1 + z \pm z^2 \pm \cdots$.

Rewriting we get

$$\begin{aligned} f(z)f(-z) &= \tfrac{1}{1-z^2} - 4z^2 \\ &= \tfrac{1-4z^2+4z^4}{1-z^2} \\ \Longrightarrow \quad (f(z)(1-z))\,(f(-z)(1+z)) &= (1-2z^2)^2 \\ \Longrightarrow \quad g(z)g(-z) &= (1-2z^2)^2 \end{aligned}$$

where $g(z) = f(z)(1 - z) = \sum b_i z^i$.

As before, $b_0 = 1$ and b_i is $0, -2$ or 2. Further, we have that $b_0 = 1, b_1 = 0, b_2 = -2$ and after that the terms alternate, taking 0 an arbitrarily long number of time (possibly empty), then 2, then 0 again an arbitrarily long number of times (possibly empty), then -2, etc.

Notice that Lemma 2.1 holds for $k \geq 2$. By looking at this equation modulo 8, we get that

$$2b_{2k} \equiv b_k^2 \pmod 8$$

This tells us that if there exists some k, $k \geq 3$ such that $b_k \neq 0$, then the minimal such k is odd.

Assume that there exists a k, odd, minimal such that $b_k = 2$. By Lemma 2.1 we have that $b_{2k} = 2$. By the alternating property of the b_i we have that there exists a j, odd, with $k < j < 2k$, j minimal such that $b_j = -2$. So we have now that $g(z) = 1 - 2z^2 + 2z^k - 2z^j + b_{j+2}z^{j+2} + \cdots$ with $g(z)g(-z) = (1-2z^2)^2$. Consider the coefficient b_{j+k} of z^{j+k}. We have from Lemma 2.1 that

$$2b_{j+k} = -2\left(\sum_{i=1}^{(j+k)/2-1} b_i b_{j+k-i}(-1)^i\right) - b_{(j+k)/2}^2$$

We see that $b_{(j+k)/2}$ is 0, because otherwise we contradict the minimality of j or k. The only term in the sum on the left hand side that is not 0 is when $i = k$. This gives us that

$$2b_{j+k} = -2b_k b_j = 8$$

The left hand side can only take the values $-2, 0$ or 2, hence a contradiction. Hence all terms $b_k = 0$ for $k \geq 3$.

Then the only (normalized) solution to this equation is when $g(x) = 1 - 2x^2$. This is equivalent to when $f(z) = 1 + z - z^2 - z^3 - z^4 - z^5 - \cdots$.

What is interesting about this solution is, if it is truncated to an even degree, then it still gives a Barker-like polynomial. For example, if $f_8(x) = 1+z-z^2-z^3-\cdots-z^8$ then $f_8(z)f_8(-z) = 1 - 3z^2 + z^4 + z^6 + \cdots + z^{16}$.

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