A LOWER BOUND FOR GARSIA'S ENTROPY FOR CERTAIN BERNOULLI CONVOLUTIONS

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Abstract. Let $\beta \in (1,2)$ be a Pisot number and let H_β denote Garsia's entropy for the Bernoulli convolution associated with β. Garsia, in 1963 showed that $H_\beta < 1$ for any Pisot β. For the Pisot numbers which satisfy $x^m = x^{m-1} + x^{m-2} + \cdots + x + 1$ (with $m \geq 2$) Garsia's entropy has been evaluated with high precision by Alexander and Zagier and later improved by Grabner Kirschenhofer and Tichy, and it proves to be close to 1. No other numerical values for H_β are known.

In the present paper we show that $H_\beta > 0.81$ for all Pisot β, and improve this lower bound for certain ranges of β. Our method is computational in nature.

1. Introduction and summary

Representations of real numbers in non-integer bases were introduced by Rényi [20] and first studied by Rényi and by Parry [17, 20]. Let β be a real number > 1. A *β-expansion* of the real number $x \in [0,1]$ is an infinite sequence of integers $(a_1, a_2, a_3, \dots)$ such that $x = \sum_{n\geq 1} a_n \beta^{-n}$. The reader is referred to Lothaire, [16, Chapter 7] for more on these topics. For the purposes of this paper, we assume that $1 < \beta < 2$ and $a_i \in \{0,1\}$.

Let μ_β denote the *Bernoulli convolution* parameterized by β on $I_\beta := [0, 1/(\beta - 1)]$, i.e.,

$$\mu_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \dots) \in \{0,1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\}$$

for any Borel set $E \subset I_\beta$, where $\mathbb{P}$ is the product measure on $\{0,1\}^{\mathbb{N}}$ with $\mathbb{P}(a_1 = 0) = \mathbb{P}(a_1 = 1) = 1/2$. Since $\beta < 2$, it is obvious that $\operatorname{supp}(\mu_\beta) = I_\beta$.

Bernoulli convolutions have been studied for decades (see, e.g., Peres, Schlag and Solomyak [18] and Solomyak [21]), but there are still many open problems in this area. The most significant property of μ_β is the fact that it is either absolutely continuous or purely singular (see Jessen and Wintner [13]); however Erdős showed that the only family of βs for which it is known to be singular, are the Pisot numbers (see [6]).

Recall that a number $\beta > 1$ is called a *Pisot number* if it is an algebraic integer whose Galois conjugates $h \neq \beta$ are less than 1 in modulus. Such is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ and, more generally, the *multinacci numbers* τ_m, the positive real root

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satisfying $x^m = x^{m-1} + x^{m-2} + \cdots + x + 1$ with $m \geq 2$. The set of Pisot numbers is typically denoted by S. It is well known that S is a closed subset of $(1, \infty)$ and moreover, the smallest Pisot number is the real cubic unit satisfying $x^3 = x + 1$ – see e.g., Bertin, Decomps, Grandet, Pathiaux and Schreiber, [3]. Amara, [2], gave a complete description of the set of all limit points of the Pisot numbers in (1,2). In particular:

Theorem 1 (Amara)**.** *The limit points of S in $(1, 2)$ are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < \psi_r < \varphi_{r+1} < \cdots < 2$$

where

$$\begin{cases} \text{the minimal polynomial of } \varphi_r \text{ is } \Phi_r(x) = x^{r+1} - 2x^r + x - 1, \\ \text{the minimal polynomial of } \psi_r \text{ is } \Psi_r(x) = x^{r+1} - x^r - \cdots - x - 1, \\ \text{the minimal polynomial of } \chi \text{ is } \mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1. \end{cases}$$

A description of the Pisot numbers approaching these limit points was given by Talmoudi [22]. Regular Pisot numbers are defined as the Pisot roots of the polynomials in Table 1. Pisot numbers that are not regular Pisot numbers are called irregular Pisot numbers. For each of these limit points (φ_r, ψ_r or χ), there exists an ϵ, (dependent on the limit point) such that all Pisot numbers in an ϵ-neighbourhood of this limit point are these regular Pisot numbers. The Pisot root of the defining polynomial approaches the limit point as n tends to infinity. It should be noted that these polynomials are not necessarily minimal, and may contain some cyclotomic factors. Also, they are only guaranteed to have a Pisot number root for sufficiently large n.

Limit Points	Defining polynomials
φ_r	$\Phi_r(x)x^n \pm (x^r - x^{r-1} + 1)$ $\Phi_r(x)x^n \pm (x^r - x + 1)$ $\Phi_r(x)x^n \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi_r(x)x^n \pm (x^{r+1} - 1)$ $\Psi_r(x)x^n \pm (x^r - 1)/(x - 1)$
χ	$\mathcal{X}(x)x^n \pm (x^3 + x^2 - x - 1)$ $\mathcal{X}(x)x^n \pm (x^4 - x^2 + 1)$

Table 1: Regular Pisot numbers

Computationally, Boyd [4, 5] has given an algorithm that will find all Pisot numbers in an interval, where, in the case of limit points, the algorithm can detect the limit points and compensate for them.

Garsia [9] introduced a new notion associated with a Bernoulli convolution. Namely, put

$$D_n(\beta) = \left\{ x \in I_\beta : x = \sum_{k=1}^{n} a_k \beta^{-k} \text{ with } a_k \in \{0, 1\} \right\}$$

and for $x \in D_n(\beta)$,

$$p_n(x) = \# \left\{ (a_1, \ldots, a_n) \in \{0, 1\}^n : x = \sum_{k=1}^{n} a_k \beta^{-k} \right\}. \tag{1}$$

Finally, put

$$H_\beta^{(n)} = -\sum_{x\in D_n(\beta)} \frac{p_n(x)}{2^n}\log\frac{p_n(x)}{2^n}$$

and

$$H_\beta = \lim_{n\to\infty}\frac{H_\beta^{(n)}}{n\log\beta}$$

(it was shown in [9] that the limit always exists). The value H_β is called *Garsia's entropy*.

Obviously, if β is transcendental or algebraic but not satisfying an algebraic equation with coefficients $\{-1,0,1\}$, then all the sums $\sum_{k=1}^n a_k\beta^{-k}$ are distinct, whence $p_n(x)=1$ for any $x\in D_n(\beta)$, and $H_\beta = \log 2/\log\beta > 1$.

However, if β is Pisot, then it was shown in [9] that $H_\beta < 1$ – which means in particular that β does satisfy an equation with coefficients $\{0,\pm 1\}$. Furthermore, Garsia also proved that if $H_\beta<1$, then μ_β is singular, but no non-Pisot β with this property has been found so far.

In 1991 Alexander and Zagier in [1] managed to evaluate H_β for the golden ratio $\beta=\tau$ with an astonishing accuracy. It turned out that H_τ is close to 1 – in fact $H_\tau\approx 0.9957$. Grabner, Kirschenhofer and Tichy [10] extended this method to the multinacci numbers; in particular, $H_{\tau_3}\approx 0.9804$, $H_{\tau_4}\approx 0.9867$, etc. They also showed that H_{τ_m} is strictly increasing for $m\ge 3$, and $H_{\tau_m}\to 1$ as $m\to\infty$ exponentially fast.

The method suggested in [1] has, however, its limitations and apparently cannot be extended to non-multinacci Pisot parameters β. Consequently, no numerical value for H_β is known for any non-multinacci Pisot β – not even a lower bound.

The main goal of this paper is to present a universal lower bound for H_β for β a Pisot number in (1,2). We prove that $H_\beta > 0.81$ for all such β (Theorem 9) and improve this bound for certain ranges of β (see discussion in Remark 7 and Proposition 10).

2. The maximal growth exponent

Denote by $\mathcal{E}_n(x;\beta)$ the set of all 0-1 words of length n which may act as prefixes of β-expansions of x. We first prove a simple characterization of this set:

Lemma 2. *We have*

$$\mathcal{E}_n(x;\beta) = \left\{(a_1,\dots,a_n)\in\{0,1\}^n \mid 0\le x-\sum_{k=1}^n a_k\beta^{-k}\le\frac{\beta^{-n}}{\beta-1}\right\}.$$

Proof. Let $(a_1,\dots,a_n)\in\mathcal{E}_n(x;\beta)$; then the fact that there exists a β-expansion of x beginning with this word, implies $\sum_1^n a_k\beta^{-k}\le x\le\sum_1^n a_k\beta^{-k}+\frac{\beta^{-n}}{\beta-1}$, the second inequality following from $\sum_{n+1}^\infty a_k\beta^{-k}\le\frac{\beta^{-n}}{\beta-1}$.

The converse follows from the fact that if $0\le y\le\frac{1}{\beta-1}$, where $y=\beta^n\big(x-\sum_{k=1}^n a_k\beta^{-k}\big)$, then y has a β-expansion $(a_{n+1},a_{n+2},\dots)$. $\square$

The following lemma will play a central role in this paper.

Lemma 3. *Suppose there exists* $\lambda \in (1,2)$ *such that* $\#\mathcal{E}_n(x;\beta) = O(\lambda^n)$ *for all* $x \in I_\beta$. *Then*

$$H_\beta \ge \log_\beta \frac{2}{\lambda}. \tag{2}$$

Proof. Let $(a_1, a_2, \dots)$ be a β-expansion of x. Denote by $p_n(a_1,\dots,a_n)$ the number of 0-1 words $(a'_1,\dots,a'_n)$ such that $\sum_{k=1}^n a_k\beta^{-k} = \sum_{k=1}^n a'_k\beta^{-k}$. Then, as was shown by Lalley [15, Theorems 1,2],

$$\sqrt[n]{p_n(a_1,\dots,a_n)} \to 2\beta^{-H_\beta}, \quad \mathbb{P}\text{-a.e. } (a_1,a_2,\dots) \in \{0,1\}^{\mathbb{N}}. \tag{3}$$

Since $p_n(a_1,\dots,a_n) \le \#\mathcal{E}_n(x;\beta)$ for $x = \sum_{k=1}^n a_k\beta^{-k}$, we have $\sqrt[n]{p_n(a_1,\dots,a_n)} \le \varepsilon_n\lambda$ with $\varepsilon_n \to 1$, which, together with (3), implies (2). ☐

Define the *maximal growth exponent* as follows:

$$\mathfrak{M}_\beta := \sup_{x\in I_\beta} \limsup_{n\to\infty} \sqrt[n]{\#\mathcal{E}_n(x;\beta)}.$$

It follows from Lemma 3 that

$$H_\beta \ge \log_\beta \frac{2}{\mathfrak{M}_\beta}. \tag{4}$$

Computing $\mathfrak{M}_\beta$ explicitly for a given Pisot β looks like a difficult problem (unless β is multinacci – see Section 6), so our goal is to obtain good upper bounds for $\mathfrak{M}_\beta$ for various ranges of β. To do that, we will need the following simple, but useful, claim.

Proposition 4. *If* $\#\mathcal{E}_{n+r}(x;\beta) \le R\cdot\#\mathcal{E}_n(x;\beta)$ *for all* $n \ge n_0$ *for some* $n_0 \ge 1$ *and some* $r \ge 2$, *then* $\mathfrak{M}_\beta \le \sqrt[r]{R}$.

Proof. By induction,

$$\#\mathcal{E}_{rn}(x;\beta) \le 2^r R^n,$$

whence follows the claim. ☐

Example 5. For the examples in this paper, we give only 4 digits of precision. In fact much higher precision was used in the computations (about 50 digits). Let us consider a toy example showing how to apply (4) to $\beta = \beta_* \approx 1.6737$, the largest root of $x^5 - 2x^4 + x^3 - x^2 + x - 1$ (which is a Pisot number).

Let us first determine $\#\mathcal{E}_2(x;\beta_*)$, dependent upon x. After that we will determine $\max_{x\in I_{\beta_*}} \#\mathcal{E}_2(x;\beta_*)$. For ease of notation, we will denote $m_n(\beta) = \max_{x\in I_\beta} \#\mathcal{E}_n(x;\beta)$. Hence in this case, we are determining $m_2(\beta_*)$. Consider the values of x such that $x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$ for initial string (a_1, a_2). We see that

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} \le x \le \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} + \dots = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{1/\beta^3}{1 - 1/\beta}.$$

This gives us upper and lower bounds for possible initial strings of (a_1, a_2).

(a_1, a_2)	Lower Bound	Upper Bound
$(0,0)$	0.	0.5300
$(0,1)$	0.3570	0.8870
$(1,0)$	0.5975	1.1275
$(1,1)$	0.9545	1.4845

Table 2: Upper and lower bounds for x for initial strings of length 2 of its β-expansion

We next partition possible values of x in $I_\beta = [0, 1.4845]$ based on these upper and lower bound.

Range (approx)	Possible initial string of expansion
$x \in (0., 0.3570)$	$(0,0)$
$x \in (0.3570, 0.5300)$	$(0,0), (0,1)$
$x \in (0.5300, 0.5975)$	$(0,1)$
$x \in (0.5975, 0.8870)$	$(0,1), (1,0)$
$x \in (0.8870, 0.9545)$	$(1,0)$
$x \in (0.9545, 1.1275)$	$(1,0), (1,1)$
$x \in (1.1275, 1.4845)$	$(1,1)$

Table 3: Initial strings (a_1, a_2), depending on $x \in (0, 1.4875)$.

This immediately shows that $m_2(\beta_*) = 2$. Hence, by induction, $\#\mathcal{E}_{n+2}(x;\beta_*) \leq 2\#\mathcal{E}_n(x;\beta_*)$, whence by Proposition 4, $\mathfrak{M}_{\beta_*} \leq \sqrt{2}$. By (4), $H_{\beta_*} > \frac{1}{2}\log_{\beta_*} 2 \approx 0.6729$.

Obviously, this bound is rather crude, and in the rest of the paper we will refine this method to obtain better bounds. One thing we need to do is show how one would use this for an entire range of β values, instead of just for a specific value. For instance, in the example above, we could show that $m_2(\beta) = 2$ for all $\beta > \tau = \frac{1+\sqrt{5}}{2}$. In addition, we will want to show how one would do this calculation for algebraic β, where we can take advantage of the algebraic nature of β.

3. The algorithm

Let us consider our toy example of $\beta = \beta_*$ again. We see that for each initial string (a_1, a_2), we got a lower and upper bound for possible $x = a_1\beta^{-1} + a_2\beta^{-2} + \dots$. For example, for $(a_1, a_2) = (1, 0)$ these were approximately 0.5975 and 1.1275 respectively. We then used these lower and upper bounds to partition I_β into ranges. We next show that if the relative order of these lower and upper bounds is not changed, then the partitioning of I_β into ranges can be done in exactly the same way.

Put $(a_1, \dots, a_k)_L = \sum_1^k a_j\beta^{-j}$ and $(a_1, \dots, a_k)_U = \sum_1^k a_j\beta^{-j} + \frac{\beta^{-k}}{\beta-1}$, i.e., $[(a_1, \dots, a_k)_L, (a_1, \dots, a_k)_U]$ is the interval of all possible values of x whose β-expansion starts with $(a_1, \dots, a_k)$. For example, $(1,0)_L = 0.5975\dots$ and $(1,0)_U = 1.1275\dots$. This says that if

$$(0,0)_L < (0,1)_L < (0,0)_U < (1,0)_L < (0,1)_U < (1,1)_L < (1,0)_U < (1,1)_U,$$

then we have

Range	Possible initial string of β-expansion of x
$x \in ((0,0)_L, (0,1)_L)$	$(0,0)$
$x \in ((0,1)_L, (0,0)_U)$	$(0,0), (0,1)$
$x \in ((0,0)_U, (1,0)_L)$	$(0,1)$
$\vdots$	$\vdots$

Table 4: Upper and lower bounds for x for initial strings of length 2 of its β-expansion

as the equivalent table to Table 2. For fixed β, these $(a_1, a_2, \dots, a_k)_L$ and $(a_1, a_2, \dots, a_k)_U$ are called *critical points for* β or simply *critical points.*

For each inequality, there are precise values of β for where the inequality will hold. For example, knowing that $\beta > 1$, we get that

$$(0,0)_U < (1,0)_L \iff \frac{\beta^{-3}}{1-\beta^{-1}} < \frac{1}{\beta} \iff \frac{1+\sqrt{5}}{2} < \beta$$

So if $\beta > \tau = 1.618\dots$ then $(0,0)_U < (1,0)_L$.

This observation means that we need to determine for which values of β we have $(a_1, a_2)_{L/U} = (a_1', a_2')_{L/U}$. We will call these values of β the *transitions points* which will affect $m_n(\beta)$.

There are some immediate observations we can make that reduces the number of equations to be checked.

- $(a_1, a_2)_L = (a_1', a_2')_L$ and $(a_1, a_2)_U = (a_1', a_2')_U$ have the same set of solutions.
- $(a_1, a_2)_L = (a_1, a_2)_U$ has no solutions.
- If $a_1 \le a_1'$ and $a_2 \le a_2'$ then none of

$$\begin{aligned} (a_1, a_2)_L &= (a_1', a_2')_L \\ (a_1, a_2)_L &= (a_1', a_2')_U \\ (a_1, a_2)_U &= (a_1', a_2')_U \end{aligned}$$

  have solutions in I_β.

The first two observations were used when finding all transition points. The last observation was made by one of the referees after all of the computations were completed, and hence was not used as a means of eliminating equations to check.

In our length 2 example again, we need to check (after elimination by the three observations above),

$$\begin{array}{llll} (0,0)_U = (0,1)_L, & (0,0)_U = (1,0)_L, & (0,0)_U = (1,1)_L & (0,1)_L = (1,0)_L, \\ (0,1)_L = (1,0)_U & (1,0)_U = (1,1)_L & (0,1)_U = (1,0)_L, & (0,1)_U = (1,1)_L. \end{array}$$

Solving all of these equations, we see that the only transition points in $(1,2)$ for length 2 are $\sqrt{2} \approx 1.4142$ and $\tau \approx 1.6180$.

So, given that we know $m_n(\beta_*) = 2$, and that we have a transition point at $\tau = 1.618\dots$, we can say for all $\beta \in (\tau, 2)$ that $m_2(\beta) = 2$. Using a similar method, we can show that for $\beta \in (\sqrt{2}, \tau)$ that $m_2(\beta) = 3$, and that for $\beta \in (1, \sqrt{2})$ that $m_2(\beta) = 4$.

It is worth noting that these results do not say what happens when $\beta = \sqrt{2}$ or $\beta = \tau$. The transition points will need to be checked separately.

There is one not so obvious, but important observation that should be made at this point. It is possible for an inequality to hold for β, where β is in a disjoint union of intervals.

For example, we have

$$(0,1,1,1,1)_L < (1,0,0,0,1)_U$$

for $\beta \in (1,\sigma) \cup (\tau,2)$, where $\sigma^3 - \sigma^2 - 1 = 0$, with $\sigma \approx 1.4656$. This means that it is possible for $m_n(\beta)$ to not be an decreasing function with respect to β. For example $m_5(1.81) = 3$, $m_5(1.85) = 4$ and $m_5(1.88) = 3$. This trend appears to become more pronounced for larger values of n.

4. Numerical computations

In this section we will talk about the specific computations, and how they were done. The process started with length $n = 2$, and then progressively worked on $n = 3,4,5,\dots$ up to $n = 14$. We used this process to find the global minimum for all $\beta \in (1.6,2)$ minus a finite set of transition points. The code for for finding transition points, numerical lower bounds, and symbolic lower bounds can be found on the homepage of the first author [11].

- For each length in order, find all solutions β to
$$(a_0,a_1,\dots,a_{n-1})_{L/U} = (a'_0,a'_1,\dots,a'_{n-1})_{L/U}$$
subject to the conditions mentioned in the previous section.
- For each of these solutions, check to see if the transition point is a Pisot number. If so, we will have to check this transition point using the methods of Section 5.
- Use these transition points to partition $(1,2)$ into subintervals, upon which $m_n(\beta)$ is constant.
- For the midpoint of each of these subintervals, compute $m_n(\beta)$,

To compute $m_n(\beta)$, we first consider all 0-1 sequences $w_1, w_2, \dots$ of length n. For each of these sequences, find their upper and lower bounds, say $\{\alpha_1,\alpha_2,\dots\} = \{w_{1L}, w_{1U}, w_{2L}, w_{2U}, \dots\}$. Here the α_i are reorder such that $\alpha_i < \alpha_{i+1}$ for all i. We then loop through each interval (α_i,α_{i+1}) and check how many of the w_i are valid on this interval. We keep track of the interval with the maximal set of valid w_i.

It should be noted that the number of times we needed to run this algorithm was rather big. At level 14, we had slightly more than $300,000$ tests where we needed to find the maximal set.

These calculations were done in Maple on 22 separate 4 CPU, 2.8 GHz machines each with 8 Gigs of RAM. These calculations were managed using the N1 Grid Engine. This cluster was capable of performing 88 simultaneous computations.

After this, we looked at all of these subintervals between transition points, and calculated the lower bounds for H_β at the endpoints, to find a global minimum. This gives rise to the main result of the paper:

Theorem 6. *If $\beta > 1.6$, and β is not a transition point for $n \le 14$, then $H_\beta > 0.81$.*

Remark 7. This theorem is weaker than necessary for most values of β. For specific ranges of values of β, we actually get a number of stronger results.

- For most $\beta \in (1.6, 2.0)$ have $H_\beta > 0.82$, (99.9 %), and a majority (51.4%) have $H_\beta > 0.87$. Here "most" is a bit misleading. Almost every β has $H_\beta = \log 2/\log \beta$. Of those that do not, there is no result that shows they should be evenly distributed, (and they most likely are not). So by "most" we mean that for some finite collection of intervals, that make up 99.9% of $(1.6, 2.0)$ that *all* β in this finite collection of intervals have $H_\beta > 0.82$.
- The minimum occurs near $\tau_3 \approx 1.8392$, (See Figure 2).
- For $\beta \in (1.6, 1.7)$ we have $H_\beta > 0.87$ (Figure 3), and for β near 2.0 we have $H_\beta > 0.9$ (Figure 4).

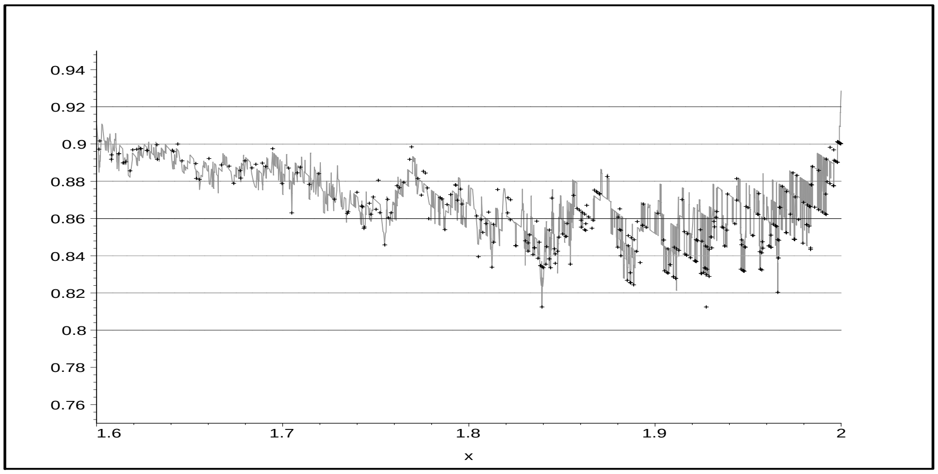

FIGURE 1. Lower bound for H_β, for Pisot $\beta \in (1.6, 2.0)$ and Pisot Transition points

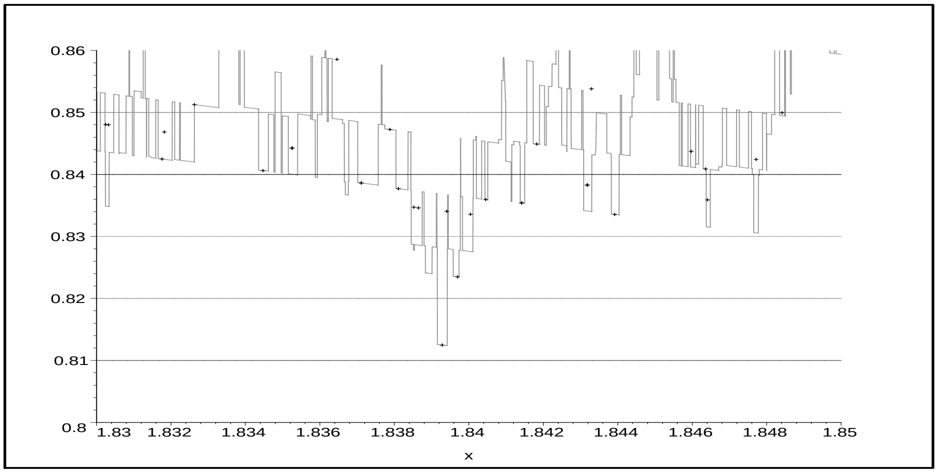

FIGURE 2. Lower bound for H_β, for Pisot $\beta \in (1.83, 1.85)$ and Pisot Transition points

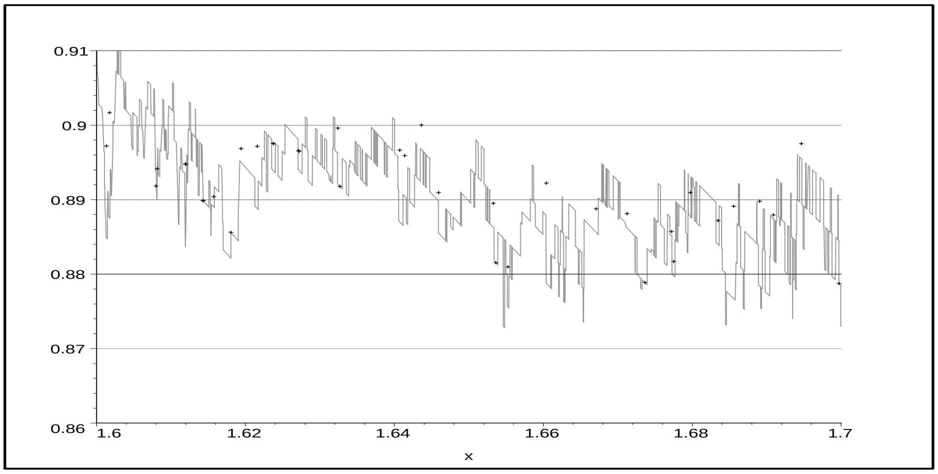

FIGURE 3. Lower bound for H_β, for Pisot $\beta \in (1.6, 1.7)$ and Pisot Transition points

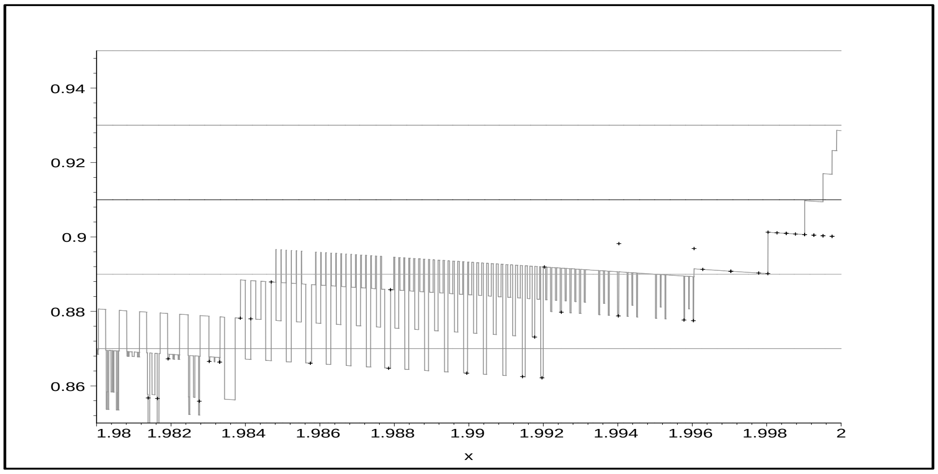

FIGURE 4. Lower bound for H_β, for Pisot $\beta \in (1.98, 2.0)$ and Pisot Transition points

5. Calculations for symbolic β

In the previous section, we showed for all but a finite number of Pisot numbers β in $(1.6, 2)$ that $H_\beta > 0.81$. To extend the result to all such β in $(1, 2)$, there are still some of Pisot numbers that will need to be checked individually.

These include the finite set of Pisot numbers less that 1.6 (of which there are 12), and the finite set of Pisot numbers that are also transition points (of which there are 427). In particular, we get:

Theorem 8. *For all Pisot numbers $\beta < 1.6$ and all Pisot transition points (for $n \leq 14$), we have $H_\beta > 0.81$.*

Combined, this theorem and Theorem 6 yield

Theorem 9. *For any Pisot* β *we have* $H_\beta > 0.81$.

Minimal polynomial of β	Pisot number	Length	Lower Bound for H_β
$x^3 - x - 1$	1.3247	17	.88219
$x^4 - x^3 - 1$	1.3803	16	.87618
$x^5 - x^4 - x^3 + x^2 - 1$	1.4433	15	.89257
$x^3 - x^2 - 1$	1.4656	15	.88755
$x^6 - x^5 - x^4 + x^2 - 1$	1.5016	14	.90307
$x^5 - x^3 - x^2 - x - 1$	1.5342	15	.89315
$x^7 - x^6 - x^5 + x^2 - 1$	1.5452	13	.90132
$x^6 - 2x^5 + x^4 - x^2 + x - 1$	1.5618	15	.90719
$x^5 - x^4 - x^2 - 1$	1.5701	15	.88883
$x^8 - x^7 - x^6 + x^2 - 1$	1.5737	14	.90326
$x^7 - x^5 - x^4 - x^3 - x^2 - x - 1$	1.5900	15	.89908
$x^9 - x^8 - x^7 + x^2 - 1$	1.5912	14	.90023

Table 5: Lower bounds for Garsia's entropy for all Pisot numbers < 1.6

As a corollary, we obtain a result on small Pisot numbers:

Proposition 10. *All Pisot* $\beta < 1.7$ *have Garsia entropy* $H_\beta > 0.87$.

There are actually a lot of advantages to doing a symbolic check as compared to the numerical techniques of the previous section. Some of these include not requiring high precision arithmetic and the combining of equivalent strings, both of which has speed and memory advantages. These are described in the example below.

To illustrate the (computer-assisted) proof of Theorem 8, consider as an example $\beta = \tau$ the golden ratio. As before, we wish to find the

$$\frac{a_1}{\tau} + \frac{a_2}{\tau^2} \le x \le \frac{a_1}{\tau} + \frac{a_2}{\tau^2} + \frac{1/\tau^3}{1 - 1/\tau}.$$

But now we can find exact symbolic values for these ranges. In particular, we notice that $\frac{1/\tau^3}{1-1/\tau} = \tau - 1$. Secondly, as $\frac{1}{\tau} = \tau - 1$ and $\frac{1}{\tau^2} = 2 - \tau$ we get

(a_1, a_2)	Lower Bound	Upper Bound
$(0,0)$	0	$\tau - 1 \approx 0.618$
$(0,1)$	$2 - \tau \approx 0.382$	1
$(1,0)$	$\tau - 1 \approx 0.618$	$2\tau - 2 \approx 1.236$
$(1,1)$	1	$\tau \approx 1.618$

Table 6: Upper and lower bounds for initial strings of length 2 for $x = a_1\tau^{-1} + a_2\tau^{-2} + \cdots$

So in particular, it *is* possible for x to start with both $(0,0)$ and $(1,0)$. But if this is the case then $x = (0,0,1,1,1,\dots) = (1,0,0,0,\dots) = \tau - 1$. So it is not possible for x to have an infinite number of expansions starting with $(0,0)$ and an infinite number of expansions starting with $(1,0)$. Similar arguments can be used for the other critical point, $x = 1$.

So we can discard the critical points and subdivide the possible values of x into the following ranges:

Range	Possible initial string of the τ-expansion
$x \in (0, 2-\tau)$	$(0,0)$
$x \in (2-\tau, \tau-1)$	$(0,0), (0,1)$
$x \in (\tau-1, 1)$	$(0,1), (1,0)$
$x \in (1, 2\tau-2)$	$(1,0), (1,1)$
$x \in (2\tau-2, \tau)$	$(1,1)$

Table 7: Initial string of τ-expansion of x, depending on x.

This immediately shows that $m_2(\tau) = 2$. Hence, by induction, $\#\mathcal{E}_{n+2}(x;\tau) \leq 2\#\mathcal{E}_n(x;\tau)$, whence $\mathfrak{M}_\tau \leq \sqrt{2}$. By (4), $H_\tau > \frac{1}{2}\log_\tau 2 = 0.7202100$.

The main advantage of this method comes when we have longer strings. In particular, it is easy to see that $(1,0,0) = (0,1,1)$. This allows us to compress information.

$a_1 a_2 a_3$	Lower Bound	Upper Bound
$(0,0,0)$	0	$5-3\tau \approx 0.1459$
$(0,0,1)$	$2\tau-3 \approx 0.2361$	$2-\tau \approx 0.3820$
$(0,1,0)$	$2-\tau \approx 0.3820$	$4-2\tau \approx 0.7639$
$(0,1,1) = (1,0,0)$	$\tau-1 \approx 0.6180$	1
$(1,0,1)$	$3\tau-4 \approx 0.8541$	$2\tau-2 \approx 1.2361$
$(1,1,0)$	1	$3-\tau \approx 1.3820$
$(1,1,1)$	$2\tau-2 \approx 1.2361$	$\tau \approx 1.6180$

Table 8: Upper and lower bounds for initial string of length 3 for $x = a_1\tau^{-1} + a_2\tau^{-2} + a_3\tau^{-3} + \dots$

This gives that for $x \in (\tau-1, 4-2\tau)$ we have the initial string of $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, and if $x \in (3\tau-4, 1)$ we have the initial string of $(1,0,1), (0,1,1), (1,0,0)$.

Our implementation does not maintain a separate entry for $(0,1,1)$ and $(1,0,0)$, as they are equivalent. Instead, the algorithm stores only one of these two strings, and indicates that this has weight 2. For the general Pisot β, this is checked by noticing that $(a_1, a_2, \dots, a_n)$ is equivalent to the same word as $(b_1, b_2, \dots, b_n)$ if and only if $a_n x^{n-1} + \dots + a_1 \equiv b_n x^{n-1} + b_{n-1}x^{n-2} + \dots + b_1 \equiv c_{d-1}x^{d-1} + \dots + c_d \pmod{p(x)}$ for some c_i, with $p(x)$ the minimal polynomial for β, of degree d. Given the large amount of overlapping that we see for large lengths, this will have major cost savings, both in memory and time.

6. THE MAXIMAL GROWTH EXPONENT FOR THE MULTINACCI FAMILY AND DISCUSSION

In this section we will compute the maximal growth exponent for the multinacci family and compare our lower bound (4) with the actual values.

Let, as above, τ_m denote the largest root of $x^m - x^{m-1} - \cdots - x - 1$ (hence $\tau = \tau_2$). Define the *local dimension* of the Bernoulli convolution μ_β as follows:

$$d_\beta(x) = \lim_{h\to 0} \frac{\log \mu_\beta(x-h, x+h)}{\log h}$$

(if the limit exists). As was shown in Lalley [15], $d_\beta(x) \equiv H_\beta$ for μ_β-a.e. $x \in I_\beta$ for any Pisot β.

Notice that it is well known that the limit in question exists if it does so along the subsequence $h = c\beta^{-n}$ for any fixed $c > 0$ (see, e.g., Feng [7]). We choose $c = (\beta-1)^{-1}$, so

$$d_\beta(x) = -\lim_{n\to\infty} \frac{1}{n} \log_\beta \mu_\beta \left(x - \frac{\beta^{-n}}{\beta-1}, x + \frac{\beta^{-n}}{\beta-1} \right). \tag{5}$$

Let $\beta = \tau_m$ for some $m \ge 2$.

Lemma 11. *Suppose β is multinacci, and put*

$$\varepsilon_\beta(x) = \lim_{n\to\infty} \sqrt[n]{\#\mathcal{E}_n(x;\beta)}.$$

Then this limit exists if and only if $d_\beta(x)$ exists, and furthermore,

$$d_\beta(x) = \log_\beta \frac{2}{\varepsilon_\beta(x)}. \tag{6}$$

Proof. Observe that the inequality on the initial terms of a β-expansion

$$0 \le x - \sum_{k=1}^{n} a_k \beta^{-k} \le \frac{\beta^{-n}}{\beta-1}$$

implies the inequality on the infinite sum

$$\sum_{k=1}^{\infty} a_k \beta^{-k} \le x + \frac{\beta^{-n}}{\beta-1} \le \sum_{k=1}^{\infty} a_k \beta^{-k} + \frac{2\beta^{-n}}{\beta-1}$$

(in view of $\sum_{n+1}^{\infty} a_k \beta^{-k} \le \beta^{-n}/(\beta-1)$), which is equivalent to

$$x - \frac{\beta^{-n}}{\beta-1} \le \sum_{k=1}^{\infty} a_k \beta^{-k} \le x + \frac{\beta^{-n}}{\beta-1}. \tag{7}$$

Hence by definition and Lemma 2,

$$\mu_\beta \left(x - \frac{\beta^{-n}}{\beta-1}, x + \frac{\beta^{-n}}{\beta-1} \right) \ge 2^{-n} \#\mathcal{E}_n(x;\beta). \tag{8}$$

Now put

$$\widetilde{\mathcal{E}}_n(x;\beta) = \left\{ (a_1, \dots, a_n) \in \{0,1\}^n \mid -\frac{\beta^{-n}}{\beta-1} \le x - \sum_{k=1}^{n} a_k \beta^{-k} \le \frac{2\beta^{-n}}{\beta-1} \right\}.$$

Since (7) implies

$$-\frac{\beta^{-n}}{\beta-1} \le x - \sum_{k=1}^{n} a_k \beta^{-k} \le \frac{2\beta^{-n}}{\beta-1},$$

we have

$$\mu_\beta \left(x - \frac{\beta^{-n}}{\beta-1}, x + \frac{\beta^{-n}}{\beta-1} \right) \le 2^{-n} \#\widetilde{\mathcal{E}}_n(x;\beta). \tag{9}$$

Combining (8) and (9), we obtain

$$2^{-n}\#\mathcal{E}_n(x;\beta) \le \mu_\beta\left(x-\frac{\beta^{-n}}{\beta-1},x+\frac{\beta^{-n}}{\beta-1}\right) \le 2^{-n}\#\widetilde{\mathcal{E}}_n(x;\beta),$$

whence

$$\begin{aligned}\log_\beta 2-\frac{1}{n}\log_\beta\#\widetilde{\mathcal{E}}_n(x;\beta) &\le -\frac{1}{n}\log_\beta\mu_\beta\left(x-\frac{\beta^{-n}}{\beta-1},x+\frac{\beta^{-n}}{\beta-1}\right)\\ &\le \log_\beta 2-\frac{1}{n}\log_\beta\#\mathcal{E}_n(x;\beta).\end{aligned} \tag{10}$$

Notice that (10) in fact holds for any β. Now we use the fact that β is multinacci. It follows from Feng, [7, Lemma 2.11] that for a multinacci β one has $\sqrt[n]{p_n(x)}\sim\sqrt[n]{p_n(x')}$ provided $|x-x'|\le C\beta^{-n}$ for any fixed $C>0$ and any $x,x'\in D_n(\beta)$ which are not endpoints of I_β. (Here $p_n(x)$ is given by (1).)

Hence $\sqrt[n]{\#\mathcal{E}_n(x;\beta)}\sim\sqrt[n]{\#\widetilde{\mathcal{E}}_n(x;\beta)}$ for all $x\in(0,\frac{1}{\beta-1})$, and (10) together with (5) yield the claim of the lemma. □

Consequently, for a multinacci β,

$$\inf_{x\in I_\beta^*} d_\beta(x)=\log_\beta\frac{2}{\mathfrak{M}_\beta}, \tag{11}$$

where $I_\beta^*=\left\{x\in(0,\frac{1}{\beta-1}):d_\beta(x)\,\text{exists}\right\}$. In [7, Theorem 1.5] it was shown that

$$\inf_{x\in I_{\tau_m}^*} d_{\tau_m}(x)=\begin{cases}\log_\tau 2-\frac{1}{2}, & m=2\\ \frac{m}{m+1}\log_{\tau_m}2, & m\ge 3.\end{cases}$$

This immediately gives us the explicit formulae for the maximal growth exponent for the multinacci family, namely,

$$\mathfrak{M}_{\tau_m}=\begin{cases}\sqrt{\tau}, & m=2\\ 2^{\frac{1}{m+1}}, & m\ge 3.\end{cases}$$

In fact, one can easily obtain the values x at which $\mathfrak{M}_\beta$ is attained. More precisely, for $\beta=\tau$ the maximum growth is attained at x with the β-expansion $(1000)^\infty$, i.e., at $x=(5+\sqrt{5})/10$.[1]

For $m\ge 3$ the maximal growth point is x with the β-expansion $(10^m)^\infty$. These claims can be easily verified via the matrix representation for $p_n(x)$ given in [7], and we leave it as an exercise for the interested reader. (Recall that the growth exponent for $p_n(x)$ is the same as for $\#\mathcal{E}_n(x;\beta)$ for the multinacci case.)

Finally, since we know the exact values of the maximal growth exponent for this family, we can assess how far our estimate (that is, the smallest value of the local dimension) is from the actual value of H_β (which is the average value of $d_\beta(x)$ for μ_β-a.e. x). Here is the comparison table:

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[1] This was essentially proved by Pushkarev [19], via multizigzag lattices techniques.

m	$\log_{\tau_m} \frac{2}{\mathfrak{M}_{\tau_m}}$	H_{τ_m}
2	0.9404	0.9957
3	0.8531	0.9804
4	0.8450	0.9869
5	0.8545	0.9926

Table 9: Lower bounds and the actual values for H_{τ_m}

We see that for $m \geq 3$ our bounds are far below H_β; moreover, our method cannot in principle produce a uniform lower bound for all β better than 0.845. However, as a first approximation it still looks pretty good.

Remark 12. We believe (11) holds for all Pisot $\beta \in (1, 2)$. If this were the case, then (4) would effectively yield a lower bound for the infimum of the local dimension of μ_β. This may prove useful, as, similarly to the entropy, no lower bound for d_β is known for the non-multinacci β. Furthermore, if one could compute the exact value of $\mathfrak{M}_\beta$, this would yield the exact value of $\inf_{x \in I_\beta^*} d_\beta(x)$.

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> In Section 6 - besides the multinacci, could you say something on $\beta = (a + \sqrt{a^2 + 4})$, with an integer $a \geq 2$? (Maybe using results from Komatsu [14].) Or, more generally, on numbers β that are root of a polynomial $X^n - a_{n-1}X^{n-1} - \ldots - a_1 X - a_0$, where $a_{n-1} \geq \ldots \geq a_1 \geq 1$?

We would also like to mention the recent paper by Feng and the second author [8], in which the average growth exponent for β-expansions is studied for the Pisot parameters β.

REFERENCES

[1] J. C. ALEXANDER and D. ZAGIER, 'The entropy of a certain infinitely convolved Bernoulli measure.' *J. London Math. Soc. (2)* 44 (1991) 121–134.

[2] M. AMARA, 'Ensembles fermés de nombres algébriques.' *Ann. Sci. École Norm. Sup. (3)*, 83 (1966) 215–270.

[3] M.-J. BERTIN, A. DECOMPS-GUILLOUX, M. GRANDET-HUGOT, M. PATHIAUX-DELEFOSSE and J.-P. SCHREIBER, *Pisot and Salem numbers* (Birkhäuser Verlag, Basel, 1992).

[4] D. W. BOYD. ' Pisot numbers in the neighborhood of a limit point. II.' *Math. Comp.*, 43 (1984) 593–602.

[5] D. W. BOYD. ' Pisot numbers in the neighbourhood of a limit point. I.' *J. Number Theory*, 21 (1985) 17–43.

[6] P. ERDŐS, 'On a family of symmetric Bernoulli convolutions.' *Amer. J. Math.* 61 (1939) 974–976.

[7] D.-J. FENG, 'The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers.' *Adv. Math.* 195 (2005) 24–101.

[8] D.-J. FENG and N. SIDOROV, 'Growth rate for beta-expansions', preprint, http://arxiv.org/abs/0902.0488.

[9] A. M. GARSIA, 'Entropy and singularity of infinite convolutions.' *Pacific J. Math.* 13 (1963) 1159–1169.

[10] P. J. Grabner, P. Kirschenhofer and R. F. Tichy, 'Combinatorial and arithmetical properties of linear numeration systems.' *Combinatorica* 22 (2002) 245–267. Special issue: Paul Erdős and his mathematics.

[11] K. G. Hare, 'Home page.' http://www.math.uwaterloo.ca/∼kghare.

[12] T.-Y. Hu, 'The local dimensions of the Bernoulli convolution associated with the golden number.' *Trans. Amer. Math. Soc.* 349 (1997) 2917–2940.

[13] B. Jessen and A. Wintner, 'Distribution functions and the Riemann zeta function.' *Trans. Amer. Math. Soc.* 38 (1935) 48–88.

[14] T. Komatsu, 'An approximation property of quadratic irrationals.' *Bull. Soc. Math. France* 130 (2002) 35–48.

[15] S. P. Lalley, 'Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution.' *J. London Math. Soc. (2)* 57 (1998) 629–654.

[16] M. Lothaire, *Algebraic combinatorics on words*, (Cambridge University Press, 2002).

[17] W. Parry, 'On the β-expansions of real numbers', *Acta Math. Acad. Sci. Hungar.* 11 (1960) 401–416.

[18] Y. Peres, W. Schlag and B. Solomyak, 'Sixty years of Bernoulli convolutions.' 'Fractal geometry and stochastics, II (Greifswald/Koserow, 1998),' (Birkhäuser, Basel, 2000), vol. 46 of *Progr. Probab.* pp. 39–65.

[19] I. A. Pushkarev, 'The ideal lattices of multizigzags and the enumeration of Fibonacci partitions.' *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 223 (1995) 280–312. English translation in *J. Math. Sci. (New York)* 87 (1997), 4157–4179.

[20] A. Rényi, 'Representations for real numbers and their ergodic properties', *Acta Math. Acad. Sci. Hungar.* 8 (1957) 477–493.

[21] B. Solomyak, 'Notes on Bernoulli convolutions.' 'Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1,' (Amer. Math. Soc., Providence, RI, 2004), vol. 72 of *Proc. Sympos. Pure Math.* pp. 207–230.

[22] F. L. Talmoudi, 'Sur les nombres de $S \cap [1, 2[$.' *C. R. Acad. Sci. Paris Sér. A-B*, 287 (1978) A739–A741.

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