BETA-EXPANSIONS FOR INFINITE FAMILIES OF PISOT AND SALEM NUMBERS

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Abstract. This paper continues the study of beta-expansions of 1 where β is a Pisot or Salem number. Special attention is given to regular Pisot numbers, and the Salem numbers that approach these Pisot numbers.

1. Introduction and Basic Definitions

While the study and representation of numbers in bases other than base 10 is quite old, the study of what happens when the base is not an integer is relatively young, at just over 50 years. The study of non-integer bases was pioneered by Rényi in 1957, [22]. In particular, he looked at such problems as uniqueness of the expansion, and ergotic properties.

We begin by formally defining a beta-expansion

Definition 1.1. *Let $\beta \in (1, 2)$. Consider the expansion*

$$x = \sum_{j=1}^{\infty} a_j \beta^{-j}$$

where $a_j \in \{0, 1\}$. Then $a_1a_2a_3 \cdots$ is a beta-expansion *for x.*

Definition 1.2. *If $a_1a_2a_3 \cdots$ is the maximal beta-expansion for x (lexicographically) then we say that $a_1a_2a_3 \cdots$ is the* greedy expansion *for x with base β. This is denoted $d_\beta(x)$.*

Algorithm 1.1 (Greedy Algorithm)**.** *Set $r_0 := x$. Set $r_n = \beta \cdot r_{n-1} \pmod 1$ and $a_n = \lfloor \beta \cdot r_{n-1} \rfloor$. Then $a_1a_2a_3 \cdots$ is the greedy expansion of* 1.

For a detailed description of the implementation of the greedy algorithm, see [1, 16].

We make a number of simplifying assumptions for this paper. First, we assume that $\beta \in (1, 2)$, although it is possible to extend this definition to any $\beta \in \mathbb{C}$ with $|\beta| > 1$, [19, Chapter 7]. Secondly, we assume that the $a_i \in \{0, 1\}$, which again can be generalized, to a larger, more complicated set. For this paper, we are only concerned with the greedy expansion, although the study of expansions other than the greedy expansion have also been looked at elsewhere [14, 15, 17]. In addition, we consider the greedy expansion of 1 as opposed to a generalized x. For the study of expansions of numbers other than 1, see [3, 4, 22].

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In this paper we are primarily concerned with greedy expansions of 1 that are either periodic or finite. If the greedy expansion of 1 is periodic or finite, then specific algebraic information about β can be recovered.

Definition 1.3. *Consider the greedy expansion* $d_\beta(1) = a_1 a_2 \cdots a_k$ *if the greedy expansion is finite, and* $d_\beta(1) = a_1 a_2 \cdots a_k (a_{k+1} \cdots a_n)^\omega$ *if the greedy expansion is eventually periodic. Define* $P_j(x) = x^j - a_1 x^{j-1} - \cdots - a_j$. *The companion polynomial is defined as:*

$$R(x) = \begin{cases} P_n(x) & \text{if } d_\beta(1) \text{ is finite} \\ P_n(x) - P_k(x) & \text{if } d_\beta(1) \text{ is eventually periodic} \end{cases}$$

Notice by the definition above that,

$$\begin{aligned} P_k(\beta) &= \beta^k - a_1\beta^{k-1} \cdots - a_k \\ &= \beta^k \left(1 - \frac{a_1}{\beta} \cdots - \frac{a_k}{\beta^k}\right) \\ &= \beta^k \left(\frac{a_{k+1}}{\beta^{k+1}} + \frac{a_{k+2}}{\beta^{k+2}} + \cdots\right) \\ &= \frac{a_{k+1}}{\beta} + \frac{a_{k+2}}{\beta^2} + \cdots \end{aligned}$$

This, combined with the periodicity or finiteness of the expansions discussed above shows that $R(\beta) = 0$. We also see that $R(x)$ is a monic integer polynomial. Hence, this implies that β is an algebraic integer. Moreover, if β has a minimal polynomial $p(x)$, then $p(x)|R(x)$. We write $R(x) = p(x)Q(x)$. The polynomial (possibly constant) $Q(x)$ is called the co-factor of the β-expansion.

A very useful combinatorial result is:

Theorem 1.1 (Parry 1960 [21])**.** *Let* $\mathbf{a} = (a_n)_{n\geq 1}$ *be a sequence in* $\{0,1\}^{\mathbb{N}_+}$. *Then the sequence* $\mathbf{a}$ *is the greedy-expansion of* 1 *for some* $\beta > 1$ *if and only if*

$$\forall\ j \geq 1, \quad \sigma^j(\mathbf{a}) <_{\text{lex}} \mathbf{a},$$

where $\sigma(a_1 a_2 a_3 \cdots) = a_2 a_3 a_4 \cdots$.

This is useful for periodic or finite expansions, as it allows us to check if a beta-expansion is greedy with a finite computation. For example, consider an eventually periodic expansion, $\mathbf{a} = a_1 a_2 \cdots a_k (a_{k+1} \cdots a_n)^\omega$. To check if this expansion is a greedy expansion, we need only check:

$$\forall\ 1 \leq j \leq n, \quad a_{j+1} a_{j+2} \cdots a_{j+n} <_{\text{lex}} a_1 a_2 \cdots a_n$$

As noted above, finite and eventually periodic greedy expansions give rise to algebraic integers. Two families of algebraic integers that are intimately related to greedy expansions are Pisot numbers and Salem numbers. To that end, define:

Definition 1.4. *A* Pisot number q *is a real algebraic integer* $q > 1$ *such that all of* q*'s conjugates are strictly less than* 1 *in modulus.*

A large amount of the structure of Pisot numbers is known. In particular, the smallest Pisot number is $1.324\cdots$, the real root of $x^3 - x - 1$ [25]. The set of Pisot numbers is known to be closed [23]. A complete description of the limit points between 1 and 2 is known, [2]. A more detailed discussion of the limit points will be given in Section 3. From a computational point of view, an algorithm to compute Pisot numbers is known [6, 8, 9]. A study of Pisot numbers with unique beta-expansion has been started [1, 16].

The next family of algebraic integers we need to discuss is the Salem numbers.

Definition 1.5. *A* Salem number α *is a real algebraic integer* $\alpha > 1$ *such that all of* α*'s conjugates are less than or equal to* 1 *in modulus, and at least one conjugate is equal to* 1 *in modulus.*

Unlike Pisot numbers, not as much is known about Salem numbers. For example, no smallest Salem number is known. The smallest found is $1.1762\cdots$, the root of $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$ [18]. Many searches have been done for small Salem numbers, see for example [7, 10, 20]. There is an important connection between Pisot numbers and Salem numbers. If $p(x)$ is the minimal polynomial of a Pisot number, then for sufficiently large m, $p(x)x^m \pm p^*(x)$ admits a Salem numbers as a root. (See for example [5].) Moreover, as $m \to \infty$, these Salem numbers approach the Pisot number as a two sided limit.

Numerous results about greedy expansions are known for Pisot and Salem numbers. The first nice result is that, if q is a Pisot number, then the greedy expansion of 1 base q is finite or eventually periodic. This is not true for a general algebraic integer. For example, consider $\beta = \sqrt{2}$. If this had a periodic or finite greedy expansion, then $x^2 - 2 | R(x)$, the companion polynomial. But, by construction of $R(x)$, the tail coefficient is ± 1, which is a contradiction.

It is also known that if α is a degree 4 Salem number, then 1 has a periodic greedy expansion base α. Boyd conjectured, and gave a heuristic arguments to support why this would be true for degree 6, but not true in general for higher degree Salem numbers [11, 12, 13].

It is not true that if 1 has a finite or periodic greedy expansion base β then β is a Pisot or Salem number. An example of this is $\beta \approx 1.7403$ the root of $x^8-x^7-x^6-x^3-x-1$, that has a finite greedy expansion $d_\beta(1) = 11001011$. A somewhat weaker result is true though [24].

Theorem 1.2 (Schmidt 1980). *If* $d_\beta(x)$ *is periodic or finite for all* $x \in \mathbb{Q} \cap [0,1)$ *then* β *is a Pisot or Salem numbers.*

Schmidt conjectured that this Theorem is in fact if and only if.

This paper continues the study of greedy expansions of 1 where β is a Pisot or a Salem number. In particular, this paper gives sufficient conditions for when a Salem number has periodic expansion, in terms of the Pisot limit that the Salem number is approaching. This paper also gives some interesting infinite families of Pisot and Salem numbers with eventually periodic or finite beta-expansions. We also show that this condition on the Pisot numbers is not so un-natural. Although it doesn't hold for all Pisot numbers, it does hold for a large number of infinite families.

2. Salem numbers approaching Pisot numbers from above

Before we give a sufficiency condition for Salem numbers, we need to give a condition for Pisot numbers, which although doesn't always occur, does occur reasonably often.

Condition 2.1. *We say a Pisot number* q *has a finite reversibly greedy (FRG) beta-expansion if*

- *$d_\beta(1) = a_1 a_2 \cdots a_k$ is finite*
- *$a_1 a_2 \cdots a_k >_{\text{lex}} a_{k-i} a_{k-i-1} a_{k-i-2} \cdots a_2$ for all $0 \le i \le k-2$.*

Pisot numbers with this property are quite common. For example, there are 3704 Pisot numbers between 1 and 2, with degree less than or equal to 30. Of these, 2420 (65.3%) of them have finite greedy expansions. Of these, 2237 (60.4% of total) satisfy Condition 2.1. Of these, all of them have reciprocal co-factors (which is also a requirement of Theorem 2.1.)

Theorem 2.1. *Let q be a Pisot number with minimal polynomial $p(x)$. Further let q have a finite reversibly greedy beta-expansion, say $a_1 a_2 \cdots a_k$. Further assume that the companion polynomial of q has co-factor polynomial $Q(x)$, which is reciprocal. Let α_m be the Salem number satisfying the polynomial $p(x)x^m + p^*(x)$. Then for $m > 2 \cdot k$ we have that the beta-expansion of α_m is*

$$\mathbf{a} = 1(a_2 a_3 .. a_k 0^{m-k-1} a_k a_{k-1} .. a_2 00)^\omega.$$

Proof. We first show that this expansion is in fact a beta-expansion of 1, in base α_m. After this, we use Theorem 1.1 and the fact that q is finite reversibly greedy to conclude that this expansion for α_m is a greedy expansion. To show the first part, we show that $p(x)x^m + p^*(x) | R(x)$, where $R(x)$ is the companion polynomial for α_m. Let $d_q(1) = a_1 a_2 .. a_k$, with $T(x)$ the companion polynomial, and $Q(x)$ the reciprocal co-factor of this expansion. By the greedy algorithm, we see that $a_1 = 1$ for all $\beta > 1$.

Let the expansion have period $M = m + k - 1$. Computing P_{M+1} and P_{2M+1} we see:

$$\begin{aligned}
(1) \qquad P_{M+1}(x) &= x^{M+1} - a_1 x^M - a_2 x^{M-1} - \cdots - a_k x^{M-k+1} \\
&\quad - a_k x^k - a_{k-1} x^{k-1} - \cdots - a_2 x^2 \\
(2) \qquad P_{2M+1}(x) &= x^{2M+1} - a_1 x^{2M} - a_2 x^{2M-1} - \cdots - a_k x^{2M-k+1} \\
&\quad - a_k x^{M+k} - a_{k-1} x^{M+k-1} - \cdots - a_2 x^{M-2} \\
&\quad - a_1 x^M - a_2 x^{M-1} - \cdots - a_k x^{M-k+1} \\
&\quad - a_k x^k - a_{k-1} x^{k-1} - \cdots - a_2 x^2
\end{aligned}$$

Subtracting equation (2) from (1) we get the companion polynomials for this expansion is:

$$\begin{aligned}
R(x) &= P_{2M+1}(x) - P_{M+1}(x) \\
&= x^{2M+1} - a_1 x^{2M} - a_2 x^{2M-1} - \cdots - a_k x^{2M-k+1} \\
&\quad - a_k x^{M+k} - a_{k-1} x^{M+k-1} - \cdots - a_2 x^{M-2} - x^{M-1} + x^M \\
&= T(x) x^{2M-k+1} + T^*(x) x^M \\
&= \left(p(x)Q(x)x^{M-k+1} + p^*(x)Q^*(x)\right) x^M \\
&= \left(p(x)Q(x)x^m + p^*(x)Q(x)\right) x^M \\
&= Q(x) \left(p(x)x^m + p^*(x)\right) x^M
\end{aligned}$$

This shows that the expansion is a valid beta-expansion for α_m. Next, it remains to show that this expansion is greedy. Letting $\mathbf{a} = 1(a_2 a_3 .. a_k 0^{m-k-1} a_k a_{k-1} .. a_2 00)^\omega$.

Limit Points	Defining polynomials
φ_r	$\Phi_A^{\pm}(x) = \Phi_r(x)x^\ell \pm (x^r - x^{r-1} + 1)$
	$\Phi_B^{\pm}(x) = \Phi_r(x)x^\ell \pm (x^r - x + 1)$
	$\Phi_C^{\pm}(x) = \Phi_r(x)x^\ell \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi_A^{\pm}(x) = \Psi_r(x)x^\ell \pm (x^{r+1} - 1)$
	$\Psi_B^{\pm}(x) = \Psi_r(x)x^\ell \pm (x^r - 1)/(x - 1)$
χ	$\mathcal{X}_A^{\pm}(x) = \mathcal{X}(x)x^\ell \pm (x^3 + x^2 - x - 1)$
	$\mathcal{X}_B^{\pm}(x) = \mathcal{X}(x)x^\ell \pm (x^4 - x^2 + 1)$

TABLE 1. Regular Pisot numbers

We first see that $\sigma^j(\mathbf{a}) <_{\text{lex}} \mathbf{a}$ for $j = 1, \cdots, k$ because $1a_2a_3\cdots a_k$ is a greedy expansion of q. Second, we see that $\sigma^j(\mathbf{a}) <_{\text{lex}} \mathbf{a}$ for $j = k+1, \cdots, m$, as $\sigma^j(\mathbf{a})$ starts with 0, and $\mathbf{a}$ starts with 1. Lastly, we see that $\sigma^j(\mathbf{a}) <_{\text{lex}} \mathbf{a}$ for $j = m+1, \cdots, M$, as $a_1a_2\cdots a_k$ is finite reversibly greedy. ☐

3. INFINITE FAMILIES OF FRG PISOT NUMBERS.

In the last section, we gave conditions for the Pisot numbers when the one sided limit of Salem numbers is eventually periodic. In this section, we show that this condition is relatively common. We do this by demonstrating a number of infinite families with this property. First though, we need to discuss in detail the structure of Pisot numbers.

Amara [2] has determined all the limit points of the Pisot numbers smaller than 2.

Theorem 3.1. *The limit points of the Pisot numbers in* $(1, 2)$ *are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < \psi_r < \varphi_{r+1} < \cdots < 2$$

where

$$(3) \qquad \begin{cases} \text{the minimal polynomial of } \varphi_r \text{ is } \Phi_r(x) = x^{r+1} - 2x^r + x - 1, \\ \text{the minimal polynomial of } \psi_r \text{ is } \Psi_r(x) = x^{r+1} - x^r - \cdots - x - 1, \\ \text{the minimal polynomial of } \chi \text{ is } \mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1. \end{cases}$$

We know exactly what families of Pisot number approach these limit points. These are called regular Pisot numbers, and are given in Table 1. These polynomials are not, in general, irreducible. They admit only one root greater than 1, (for sufficiently large m), which is the regular Pisot number. Any other factors of these polynomials are always cyclotomic. (As an example, $\Phi_A^+(x)$ always has a factor of $(x - 1)$.)

The set of Pisot numbers that are not regular are called irregular. It is know that for any $\epsilon > 0$ that $[1, 2-\epsilon]$ contains only finitely many irregular Pisot numbers.

Table 2 contains the greedy expansion of some of these limit points and regular Pisot numbers. It also indicates if these greedy expansions satisfy the FRG condition. Lastly, Table 2 gives a pseudo co-factor associated with each expansions. This is such that $R(x)$, the companion polynomial, can be written as $R(x) = P(x)Q(x)$, where $P(x)$ is the (not necessarily minimal) defining polynomial from Table 1, or equation (3). It should be commented that we only did some of the regular Pisot numbers, as the general expansions of general regular Pisot numbers proved to be too complicated to formulate a pattern. (This, in theory could be done but was deemed uninteresting, as only the existence of infinite families was in question.) It is worth observing that experimentally $\Phi_A^-(x), \Phi_C^+(x), \Psi_B^-(x)$ and $\Psi_B^+(x)$ appear to always satisfy the FRG condition, regardless of the restriction. This is based on a check of all polynomials with $r, \ell \leq 50$ that admitted a Pisot number of this form between 1 and 2.

The proof that all of these expansions are as stated, along with what their pseudo co-factors are, is exactly the same. For this reason, we only give one as an example and leave the others as an exercise to the interested reader.

Proof of $\Phi_A^-(x)$. We see that $\Phi_A^-(x)$ satisfies

$$\Phi_A^-(x) = \left(x^{r+1} - 2x^r + x - 1\right) x^n - x^r + x^{r-1} - 1.$$

We see that the companion polynomial of the greedy expansion is

$$R(x) = x^{n+2r} - \left(\left(\left(\frac{x^r}{x-1} - (x-1)^{-1}\right) x^{r-1}x + 1\right) x^{n-2r}x + 1\right) x^r x^{r-1} - \frac{x^{r-1}}{x-1} + (x-1)^{-1}.$$

Letting the pseudo co-factor be:

$$C(x) = \frac{x^r - 1}{x - 1} = x^{r-1} + x^{r-2} + \cdots + x + x,$$

it is a simple algebraic check to see that $R(x) = \Phi_A^-(x)C(x)$.

The fact that this is a greedy expansion follows directly from Theorem 1.1 and the fact that the expansion starts with r consecutive 1's, which is also the largest set of consecutive sequence of 1's in the sequence. $\square$

4. SALEM NUMBERS APPROACHING ϕ_r AND ψ_r FROM BELOW

Now we have examined Salem numbers which approach certain Pisot numbers from above. We now examine Salem numbers which similarly approach Pisot numbers from below. In this case, we are unable to find a Theorem like that of Theorem 2.1 which gives a general expansion in terms of the greedy expansion of the Pisot number. It will become apparent why when looking at the expansions of some specific examples. So first start out with the Pisot numbers defined by

$$\Phi_r(x) = x^{r+1} - 2x^r + x - 1, \quad r \geq 1$$

and similarly take

$$q_{r,k}(x) = x^k \cdot \Phi_r(x) - \Phi_r^*(x), \quad k \geq 1$$

to define the Salem numbers we are interested in. One of the things that happens when the minus sign is introduced is that there is an obvious reflective property to these polynomials.

Lemma 4.1. *We have* $q_{r,k}(x) = q_{k,r}(x)$.

Proof. To see this, consider

$$\begin{aligned} q_{r,k}(x) &= x^k \cdot (x^{r+1} - 2 \cdot x^r + x - 1) + x^{r+1} - x^r + 2 \cdot x - 1 \\ &= x^{r+k+1} - 2 \cdot x^{r+k} + x^{k+1} - x^k + x^{r+1} - x^r + 2 \cdot x - 1 \\ q_{k,r} &= x^{r+k+1} - 2 \cdot x^{r+k} + x^{r+1} - x^r + x^{k+1} - x^k + 2 \cdot x - 1 \end{aligned}$$

So there is a symmetry not evident in the case where the Salem numbers approach a sequence of Pisot numbers from above. $\square$

Because of this symmetry, it suffice to show what happens when $r \geq k$.

Theorem 4.1. *Let* $q_{r,k}(x) = x^k \cdot \Phi_r(x) - \Phi_r^*(x)$ *Let* $r > k$ *and* $k \geq 3$. *Let* $r = m + 2k\ell$, $1 \leq m \leq 2k$. *Then*

- *If* $m = 0$ *then the greedy expansion is*
$$1\left(\left(1^{k-1}0^k1\right)^{\ell-1}1^{k-2}01^{k-1}01^{k-2}\left(10^k1^{k-1}\right)^{\ell-1}00\right)^{\omega}$$
- *If* $m = 1, 2, 3, \cdots, k-1$ *then the greedy expansion is*
$$1\left(\left(1^{k-1}0^k1\right)^{\ell-1}1^{k-1}0^k1^{m-1}01^{k-m-1}01^{m-1}\left(0^k1^k\right)^{\ell-1}0^k1^{k-1}00\right)^{\omega}$$
- *If* $m = k+1, k+2 \cdots, 2k-1$ *then the greedy expansion is*
$$1\left(\left(1^{k-1}0^k1\right)^{\ell}1^{k-2}01^{m-k-1}01^{k-2}\left(10^k1^{k-1}\right)^{\ell}00\right)^{\omega}$$

Proof. It suffices to notice that the pseudo co-factors are: $\frac{x^{k(2\,\ell+1)}+1}{(x^k+1)(x-1)}$, $\frac{x^{k(2\,\ell-1)}+1}{(x^k+1)(x-1)}$, and $\left(-\frac{\left(-x^k\right)^{2\,\ell+1}}{x^k+1} + \left(x^k+1\right)^{-1}\right)(x-1)^{-1}$, respectively. ☐

5. Comments and Questions

This paper helps explain part of the structure of the beta-expansions where base is a Pisot or Salem number. It is probably doable to completely determine the beta-expansion of all regular Pisot numbers. It should also be notice that Theorem 4.1 is only defined for $m = 0, 1, \cdots, k-1, k+1, \cdots, 2k-1$. The case when $m = k$ is missing. Even in the simple case, of $k = 3, r = 9$, (which gives $m = k$), this does not appear to have a periodic expansion. (If it does have a periodic expansion, the period is greater than 500,000.) It is still unknown if Salem numbers can have non-periodic expansions, although this is conjectured to be true by Boyd.

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Polynomial	Expansion	Restrictions	Comments	Pseudo Co-Factor
Limit Points				
$\Phi_r(x)$	$1^r 0^{r-1} 1$	none	FRG	$x^{r-1} + \cdots + 1$
$\Psi_r(x)$	1^{r+1}	none	FRG	1
$\mathcal{X}(x)$	$11(10)^\omega$	none		NA
Regular Pisot Approaching ϕ_r				
$\Phi_A^-(x)$	$1^r 0^{r-1} 1 0^{\ell-2r} 1 0^r 1^{r-1}$	$\ell \geq 2r-1$	FRG	$x^{r-1} + .. + 1$
$\Phi_B^-(x)$	$1^r 0^{r-1} 1 0^{\ell-2r} 1^{r-1} 0^r 1$	$\ell \geq 2r-1$	FRG	$x^{r-1} + .. + 1$
$\Phi_C^-(x)$	$1^r 0^{r-1} 1 (0^{\ell-2r} 1^r 0^r)^\omega$	$\ell \geq 2r-1$		NA
$\Phi_A^+(x)$	$(1^r 0^r)^k 1^{r-1}$	$\ell = 2rk + r - 1$	FRG	$1/(x^{r+1} - x^r + x - 1)$
$\Phi_B^+(x)$	$(1^r 0^r)^k 1$	$\ell = 2rk + 1$	FRG	$1/(x^{r+1} - x^r - x + 1)$
$\Phi_C^+(x)$	$(1^r 0^r)^{k-1} 0^{\ell-1} 1$	$\ell = 2rk$	FRG	$(1 + x + x^2 .. + x^{r-1})$ $(1 + x^{2r} + x^{4r} + .. + x^{(k-1)2r})$
Regular Pisot Approaching ψ_r				
$\Psi_A^-(x)$	$1^{r+1} (0^{\ell-r-1} 1^r 0)^\omega$	$\ell \geq r$		x^n
$\Psi_B^-(x)$	$1^{r+1} 0^{\ell-r} 1^r$	$\ell \geq r-1$	FRG	1
$\Psi_A^+(x)$	$(1^r 0)^k 0^{\ell-1} 1$	$\ell = (r+1)k$	FRG	$1 + x^{r+1} + .. + x^{(r+1)(k-1)})$
$\Psi_B^+(x)$	$(1^r 0)^{k-1} 1^r$	$\ell = (r+1)k + r$	FRG	$1/(x^{r+1} - 1)$
Regular Pisot Approaching χ				
$\mathcal{X}_A^+(x)$	$11(10)^{k-1}01000(10)^{k-1}0(00)^k 11$	$\ell = 2k+1, k \geq 2$	FRG	NA
$\mathcal{X}_A^+(x)$	$11(10)^{k-2}0111000(10)^{k-3}000010(00)^{k-2}11$	$\ell = 2k, k \geq 2$	FRG	$1 + x^2 + \cdots + x^{2k})(1 + x^{2k-1})$
$\mathcal{X}_A^-(x)$	$11(10)^{k-2}11(00011(10)^{k-2}00)^\omega$	$\ell = 2k+1, k \geq 3$		NA
$\mathcal{X}_A^-(x)$	$11(10)^{k-2}11011((10)^{k-2}0111(01)^{k-2}1000)^\omega$	$\ell = 2k, k \geq 1$		NA
$\mathcal{X}_B^+(x)$	$11(10)^{k-1}001$	$\ell = 2k+1, k \geq 3$	FRG	$x^2 - 1$
$\mathcal{X}_B^+(x)$	$11(10)^{k-2}0101(1(10)^{k-3}(011)^2(10)^{k-3}010^4 100)^\omega$	$\ell = 2k, k \geq 2$		NA
$\mathcal{X}_B^-(x)$	$11(10)^{k-2}1101000(10)^{k-3}011(1(00)^{k-1}10)^\omega$	$\ell = 2k+1, k \geq 3$		NA
$\mathcal{X}_B^-(x)$	$11(10)^{k-3}1100000(10)^{k-3}001$	$\ell = 2k, k \geq 2$	FRG	$1/(1 + x^2 + \cdots + x^{2k-4})$

TABLE 2. Expansions for Regular Pisot Numbers