BETA-EXPANSIONS OF PISOT AND SALEM NUMBERS

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Abstract. This paper is based on a talk given at WWCA (Waterloo Workshop on Computer Algebra) held at Wilfird Laurier University, April 2006. This paper gives a history of beta-expansions, and surveys some of the computational aspects of beta-expansions. Special attention is given to how these beta-expansions relate to Pisot and Salem numbers. This paper also gives an overview of the computational issues that arose in the recent investigation of Allouche, Frougny and Hare, [2], upon which the talk at WWCA was based.

1. Introduction and History

People have represented numbers in a varieties of different bases. By far the most common today is base 10, the current system inherited from the Arabics. Some other bases that have been used historically include the base 60 (mixed representation) of the Babylonians, and the base 20 (mixed representation) of the Mayans [27]. More recently, base 2 and 16 have become more common, with modern day computer science.

Let us consider the standard base 10 representation for now. Assume for convenience that $x \in [0, 1)$. We say

$$x = 0.a_1a_2a_3 \cdots = \sum_{i=1}^{\infty} \frac{a_i}{10^i}. \tag{1}$$

Here the $a_i \in \{0, 1, \cdots, 9\}$. The middle expression of eq. (1) is called a base 10 representation of x. We say a representation of an $x \in [0, 1)$ is *eventually periodic* if it can be written as

$$x = 0.a_1a_2 \cdots a_k(a_{k+1}a_{k+2} \cdots a_n)^\omega.$$

If we assume further than $a_k \neq a_n$ and that the period has minimal length, then we say that the representation has a *pre-period* of length k and a *period* of length $(n - k)$. If the period is length 1 and $a_n = 0$, then we say that the representation is *finite* and we omit the 0^ω in the representation to get $x = 0.a_1a_2 \cdots a_k$.

An elementary result about decimal numbers is that the representation of a number is eventually periodic or finite if and only if it represents a rational number. It is easy to see that this is true for any base $q \in \mathbb{Z}$, with $q \geq 2$. An immediate question present itself. What happens if $q \notin \mathbb{Z}$?

Rényi [37], was among the first to study representations of a value x with a non-integer base β. These have come to be known as beta-expansions. To this end we define the beta-expansion of a number x.

2000 *Mathematics Subject Classification.* Primary 11R06, Secondary 11A67.
Key words and phrases. Beta-expansions, Pisot numbers, Salem numbers, Univoque.
Research of K. G. Hare supported, in part by NSERC of Canada.

Definition 1.1. *Let* $x \in \left[0, \frac{\lceil\beta\rceil - 1}{\beta - 1}\right)$ *satisfy*

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}$$

where $a_n \in \{0, 1, \cdots, \lceil\beta\rceil - 1\}$*. Then* $a_1 a_2 a_3 \cdots$ *is a* **beta-expansion** *for* x*.*

For the purposes of this paper, β is always a real non-integer, and q is always an integer. Furthermore, we always assume that β and q are strictly greater than 1. Note, for convenience, we omit the "0." in front of $a_1 a_2 \cdots$. This is done first, to be consistent with the literature on beta-expansions, and secondly, because it is more natural to think of a beta-expansion as an infinite word, for which a number of the results have a more elegant description.

We now consider our first example with a non-integer base.

Example 1. *Consider* β *the golden ratio,* $\beta \approx 1.618$*, the larger root of* $\beta^2 - \beta - 1 = 0$*.*
Then

$$\begin{array}{rcll} 1 & = & \frac{1}{\beta} + \frac{1}{\beta^2} & \text{or} \\ & = & \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^4} & \text{or} \\ & = & \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{1}{\beta^5} + \frac{1}{\beta^7} + \frac{1}{\beta^9} + \cdots . & \end{array}$$

So the beta-expansions of 1 *include*

$$110^\omega = 11, \quad 1011, \quad \text{and} \quad (10)^\omega.$$

As we can see from this example, it is possible to have multiple beta-expansions for the same number. In fact, in the case of the golden ratio, 1 has an infinite number of beta-expansions. The number of expansions of 1 has been studied in [21], where they showed that for all $1 \leq N \leq \omega$ there exists 2^ω real numbers $\beta \in (1, 2)$ for which 1 has exactly N different beta-expansions.

Problems relating to a the structure of a general beta-expansion can be found in [26]. In particular, they related the properties of these beta-expansions to particular properties of the spectrum of β defined as

$$Y(\beta) = \left\{ \sum_{i=0}^{k} \beta^{n_i} : k \in \mathbb{N}, n_i \in \mathbb{N}, n_i < n_{i+1} \right\} = \{0 = y_0 < y_1 < y_2 < \cdots\}.$$

The two specific constants $\ell(\beta) = \liminf(y_n - y_{n-1})$ and $L(\beta) = \limsup(y_n - y_{n-1})$ related to this spectrum have received much attention recently. See for example [9, 10, 23, 25].

If we have multiple beta-expansions for a value, then we can order these beta-expansions lexigraphically. Of particular interest are the largest and the smallest beta-expansions lexigraphically. We begin with the largest, which has received the most attention in the literature. This is called the *greedy expansion.* (The name is suggested by the fact that the algorithm used to compute this expansion is a greedy algorithm.)

Definition 1.2. *If* $d_\beta(x) = a_1 a_2 a_3 \cdots$ *is the maximal beta-expansion for* x *(lexigraphically) then we say that* $a_1 a_2 a_3 \cdots$ *is the* **greedy expansion** *for* x *with base* β*.*

It should be noted that in some literature when they talk about the beta-expansion of x, they mean the greedy expansion of x. To avoid confusion, we will always specify an expansion as a greedy expansion or a lazy expansion if it is one of these expansions. In this paper, when we say beta-expansion we are not assuming that it is the greedy expansion, but instead that it is a general expansion of the form given in Definition 1.1.

The beta-expansions that Rényi studied were these greedy expansions. The majority of his paper was studying more general expansions such as the continued fraction expansion, or the regular q-adic expansion, where $q \in \mathbb{Z}$, with $q \geq 2$.

Rényi studied expansions of the form

$$x = \epsilon_0 + f(\epsilon_1 + f(\epsilon_2 + \cdots) \cdots)$$

with digits ϵ_n. In the case of the q-adic expansion we have $f(x) = x/q$, and in the case of the continued fraction expansion we have $f(x) = 1/x$. We can compute the digits by letting $\phi(x) = q \cdot x$ (or $\phi(x) = 1/x$ respectively) be the inverse of $f(x)$, and defining $r_n(x) = \phi(r_{n-1}(x)) \mod 1$. Then the digits are computed by $\epsilon_n = \lfloor \phi(r_{n-1}(x)) \rfloor$. Rényi showed that for an algorithm such as the continued fraction, or q-adic expansion algorithm, the digits are independent. He then considered the obvious extension of the q-adic expansion to the greedy expansion given by $f(x) = x/\beta$ with $\beta \in \mathbb{R} \setminus \mathbb{Z}$, $\beta > 1$ and $\phi(x) = \beta \cdot x \mod 1$. He observed that for the greedy expansion, the digits were not independent. An easy example of this is the case of $\beta = 1.618 \cdots$ the greater root of $x^2 - x - 1$. The greedy expansion cannot contain as a substring "011". Assume to the contrary that the greedy expansion for some x contains the substring "011". If we replace the substring "011" with "100" then we get an equally valid beta-expansion for the same number, but the latter is lexigraphically bigger, contradicting the assumption that the original beta-expansion was the greedy expansion.

Algorithmically, it is straightforward to compute the greedy expansion. If the greedy expansion is eventually periodic, then it is possible to detect this, and give the complete expansion. If the greedy expansion is not eventually periodic, then it is possible to compute any number of terms of the expansion, relatively quickly.

Algorithm 1.1 (Greedy Algorithm). *Set $r_0 := x$. Set $r_n = \beta \cdot r_{n-1} \pmod 1$ and $a_n = \lfloor \beta \cdot r_{n-1} \rfloor$. Then $a_1 a_2 a_3 \cdots$ is the greedy expansion of x.*

Some implementation issues are discussed in Section 3. Some examples of greedy expansions are given in Table 1.

We see that one of the key computations in Rényi's study (and the greedy algorithm) is $r_n(x) = \phi(r_{n-1}(x)) \pmod 1$. From this we define the operator $T_\beta(x)$ as $T_\beta(x) = \beta \cdot x \pmod 1$. We see that a necessary and sufficient condition for the greedy expansion being periodic or finite is that $T_\beta^{(n)}(x)$ is eventually periodic or eventually 0. To that end we define

Definition 1.3. *We define* $\mathrm{Fin}(\beta)$ *as the set of all x such that $T_\beta^{(n)}(x)$ is eventually 0.*

Definition 1.4. *We define* $\mathrm{Per}(\beta)$ *as the set of all x such that $T_\beta^{(n)}(x)$ is eventually periodic.*

We see that if $T_\beta^{(n)}(x)$ is eventually 0, then it is eventually periodic, with a period of length 1, and period "0". Hence $\mathrm{Fin}(\beta) \subset \mathrm{Per}(\beta)$. We notice that if β is

x	Greedy Expansions	Lazy Expansions
	$\beta \approx 1.618$ the root of $\beta^2 - \beta - 1$	
1/5	$(0001001010100100100 0)^\omega$	$000(01101^6 0110101011)^\omega$
2/5	$(010^6 100101010010)^\omega$	$0(010101101101^6 011)^\omega$
3/5	$(0101010010010^6 10)^\omega$	$0(01^6 0110101011011)^\omega$
4/5	$(10010010^6 1001010)^\omega$	$0(110110101011011011 11)^\omega$
1	11	$0(1)^\omega$
	$\beta \approx 1.325$ the root of $\beta^3 - \beta - 1$	
1/5	$(0^5 10^8 10^9)^\omega$	$0^9(1101111011110 1^5 01^5)^\omega$
2/5	$(00010^5 1000010^9)^\omega$	$0^7(1^5 01^6 01^5 01^5)^\omega$
3/5	$(010^{10} 10^6 10000)^\omega$	$0^5 10111(111101^{12} 01^5 0)^\omega$
4/5	$1000(0^7 100001000010^5 1)^\omega$	$0000(101^8 01^{13})^\omega$
1	10001	$0000(1)^\omega$
	$\beta = 2$	
1/5	$(0011)^\omega$	$(0011)^\omega$
2/5	$(0110)^\omega$	$(0110)^\omega$
3/5	$(1001)^\omega$	$(1001)^\omega$
4/5	$(1100)^\omega$	$(1100)^\omega$
1	$(1)^\omega$	$(1)^\omega$

Table 1: Table of some greedy and lazy expansions

an integer, then $\mathrm{Fin}(\beta)$ is exactly the set of numbers $\frac{n}{\beta^k}$ with $n \in \mathbb{Z}$ and $\mathrm{Per}(\beta)$ is the rationals.

One interesting difference between $\mathrm{Fin}(\beta)$ and $\mathrm{Per}(\beta)$ from when β is an integer, and when β is a general real number, is that when β is an integer, both of $\mathrm{Fin}(\beta)$ and $\mathrm{Per}(\beta)$ are necessarily closed under addition and multiplication. Moreover tight bounds upon the length of the fractional part can be given, based on the lengths of the fractional parts of the two terms to be added or multiplied. It is not necessarily true that these sets are closed under addition or multiplication for a general real number β. For some specific real numbers β with special algebraic properties, this can still be done. See for example [6] for results and software.

An obvious question occurs. What do $\mathrm{Fin}(\beta)$ and $\mathrm{Per}(\beta)$ look like in general, when β is not an integer? When can we say $\mathrm{Per}(\beta)$ contains $\mathbb{Q}$?

Before discussing this, we need to introduce some standard definitions from algebraic number theory.

Definition 1.5. *A number α is an* **algebraic integer** *if it is the root of a monic integer polynomial. There is a unique monic integer polynomial $p(x)$, called the* **minimal polynomial***, for which α is a root and the degree of $p(x)$ is minimal.*

Definition 1.6. *If α is an algebraic integer, and $p(x)$ is its minimal polynomial, then we say that all of the other roots of $p(x)$ are the* **conjugates** *of α.*

Definition 1.7. *A* **Pisot number** *α is a real algebraic integer $\alpha > 1$ such that all of α's conjugates are strictly less than 1 in modulus.*

Definition 1.8. *A* **Salem number** α *is a real algebraic integer* $\alpha > 1$ *such that all of* α*'s conjugates are less than or equal to 1 in modulus, and at least one conjugate is equal to 1 in modulus.*

Example 2. *An example of an algebraic integer is* $1 + \sqrt{2} + \sqrt{3}$ *with minimal polynomial* $x^4 - 4x^3 - 4x^2 + 16x - 8$*. The conjugates of* $1 + \sqrt{2} + \sqrt{3}$ *are* $1 \pm \sqrt{2} \pm \sqrt{3}$*.*

An example of a Pisot number is, $1.325\cdots$*, the root of* $x^3 - x - 1$*. The two conjugates of this Pisot number are both of modulus* $0.8689\cdots < 1$*.*

An example of a Salem number is $1.7221\cdots$*, the root of* $x^4 - x^3 - x^2 - x + 1$*, the other conjugates have modulus* $1, 1$ *and* $0.5807\cdots$*.*

The structure of the set of all Pisot numbers is well understood. The set is known to be closed [38], with a smallest value of $1.324\cdots$, the real root of $x^3 - x - 1$ [40]. Amara gave a complete description of the set of all limit points of the Pisot numbers [3]. Boyd has given an algorithm that will find all Pisot numbers in an interval, where, in the case of limit points, the algorithm can detect the limit points and compensate for them [12, 13]. Both of these results were exploited for the computational exploration of beta-expansions of Pisot numbers given in [2], (see Section 2 for more details).

The set of Salem numbers is not as well understood. It is known that every Pisot number is the limit of Salem numbers from both above and below. No smallest value in the set of Salem numbers is known. In fact, a major open conjecture is if there is a smallest Salem number, and if so, what is it [8]? The smallest known Salem number is $1.1762\cdots$, the root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, which was found in 1933 by Lehmer [33]. Despite numerous computer searches since then, no better example has been found [11, 14, 35].

So, now that we have some of the basic tools, we can talk about what $\mathrm{Fin}(\beta)$ and $\mathrm{Per}(\beta)$ look like when β is a non-integer. In [7], it is shown that if β is a Pisot number then a necessary and sufficient condition for x to have an eventually periodic beta-expansion, is that $x \in \mathbb{Q}(\beta)$. Schmidt [39] showed that $\mathbb{Q} \cap [0, 1) \subset \mathrm{Per}(\beta)$ implies that β is a Pisot or a Salem number. Moreover, he showed that if β is not a Pisot nor a Salem number, then $\mathrm{Per}(\beta) \cap \mathbb{Q}$ is nowhere dense in $[0, 1)$. An important conjecture, that has motivated a lot of later work is that

Conjecture (Schmidt's Conjecture). *We have* $\mathbb{Q} \cap [0, 1) \subset \mathrm{Per}(\beta)$ *if and only if* β *is a Pisot or a Salem number.*

One direction is known already, namely that $[0, 1) \cap \mathbb{Q} \subset \mathrm{Per}(\beta) \cup \mathrm{Fin}(\beta)$ implies that β is a Salem or Pisot number. The other direction is shown only for the Pisot number case. All that remains to show is that β being a Salem number implies that $\mathbb{Q} \cap [0, 1) \subset \mathrm{Per}(\beta)$.

A related question to this, first investigated by Parry [36], concerns when $1 \in \mathrm{Per}(\beta)$. Another way to phrasing this is determining for which β do we have $d_\beta(1)$ is either eventually periodic or finite. We say that β is a beta-number if $d_\beta(1)$ is eventually periodic, and we say that β is a simple beta-number if $d_\beta(1)$ is finite. Much like the situation with Schmidt's conjecture, it is known that if β is a Pisot number, then β is a beta-number. A very simple argument of Boyd's shows that if β is a Salem number, then $d_\beta(1)$ cannot be finite, and hence cannot be a simple beta-number [14]. But this raised the question, how often can Salem numbers be beta-numbers?

Motivated by this Boyd looks at the greedy expansion of 1. In [14], Boyd shows that if β is a Salem number of degree 4 then $1 \in \mathrm{Per}(\beta)$. In fact, he proved if β is a degree 4 Salem number, then $d_\beta(1)$ had a pre-period of length 1 and a period of length 3, 5, 9 or an even number. Furthermore, the length of the period is bounded by $2\beta + 3$. He gave heuristical evidence to show that if β is a Salem number then β would be a beta-number for degree six, but for higher degrees, β would not be a beta-number a positive proposition of the time.

Consider the expansion $d_\beta(1) = a_1 a_2 \cdots a_k$ if the greedy expansion is finite, and $d_\beta(1) = a_1 a_2 \cdots a_k (a_{k+1} \cdots a_n)^\omega$ if the greedy expansion is eventually periodic. Define $P_i(x) = x^i - a_1 x^{i-1} - \cdots - a_i$. We define the companion polynomial as:

$$R(x) = \begin{cases} P_n(x) & \text{if } d_\beta(1) \text{ is finite} \\ P_n(x) - P_k(x) & \text{if } d_\beta(1) \text{ is eventually periodic} \end{cases}$$

By considering β as a beta-number (or simple beta-number), we see that $R(x)$ is a well defined monic integer polynomial. Moreover $R(\beta) = 0$. Thus we see that β must be an algebraic integer. Let $p(x)$ be the minimal polynomial for β, then we see that $p(x)|R(x)$. In fact we can write $R(x) = p(x)Q(x)$. Here $Q(x)$ is called the co-factor of the beta-expansion, where as $R(x)$ is called the companion polynomial.

An equivalent analysis can be done for $d_\beta(\alpha)$ by defining $P_i(x) = \alpha x^i - a_1 x^{i-1} - \cdots - a_i$. To complete this analysis we would have to look at the algebraic properties of α as well as β. This means we may not be able to restrict our attention to algebraic integers. For the purposes of this paper, we assume that we are considering the greedy expansion of 1, and hence $\alpha = 1$.

The study of the companion polynomial, such as the location of its roots was initiated by Parry. In particular Parry showed the roots of $R(x)$, other than β are in $|z| < \min(2, \beta)$, [36]. This was improved to $|z| \leq \frac{1+\sqrt{5}}{2}$ by Solomyak [41] and independently by Flatto, Lagarias, and Poonen [22].

Boyd proved that the co-factor is reciprocal and cyclotomic when β is a degree 4 Salem numbers. He also gave heuristics to show that this is true for degree 4 and 6, but not true for a positive proportion of degree 8 or more ([15, 17]). In [15] Boyd did a massive computation to check the degree 6 case. He checked the co-factors of 11836 degree 6 Salem numbers (of reasonably small trace).

Based on preliminary experimental evidence, it was conjectured that the complementary factor was always cyclotomic (and hence reciprocal). Boyd [15] showed that this is false for Pisot numbers, by doing a large search over a particular set of Pisot numbers. In particular, he looked at:

- All Pisot numbers up to degree 50 in [1.9, 2]
- All Pisot numbers up to degree 60 in [1.96, 2]
- All Pisot numbers up to degree 20 in [2, 2.2]
- All irregular Pisot numbers in $[1, 1.9324] \cup [1.9333, 1.96]$
- Pisot numbers associated with the first 8 limit point.

And he found examples of Pisot numbers whose co-factors were both non-reciprocal as well as reciprocal but non-cyclotomic. He also found infinite families of examples of each of these.

Most of our discussion so far has been derived from properties of the greedy expansion (the maximal expansion lexigraphically.) At this point, we need to introduce the other important beta-expansion. Namely, the minimal beta-expansion lexigraphically, the *lazy expansion*.

Definition 1.9. *If $\ell_\beta(x) = a_1a_2a_3\cdots$ is the minimal beta-expansion for x (lexicographically) then we say that $a_1a_2a_3\cdots$ is a* **lazy expansion***.*

Algorithmically, the lazy expansion is easy to compute, with the same caveat as before that if the expansion is not eventually periodic or finite, then the best we can hope for is to be able to compute a large number of terms.

Algorithm 1.2 (Lazy Algorithm). *Consider a sequence*

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_k}{\beta^k} + \frac{?}{\beta^{k+1}} + \cdots$$

where we wish to determine if a_{k+1} should be 0 or 1, in the lazy expansion. If

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_k}{\beta^k} + \frac{0}{\beta^{k+1}} + \frac{1}{\beta^{k+2}} + \frac{1}{\beta^{k+3}} + \frac{1}{\beta^{k+4}} + \cdots \geq \alpha$$

then $a_{k+1} = 0$ as we don't "need" $a_{k+1} = 1$. If this sum is $< \alpha$, then we "need" $a_{k+1} = 1$. The expansion is denoted by $\ell_\beta(\alpha)$.

Implementation issues are discussed in Section 3. Some examples of lazy expansions are given in Table 1.

There has been a study of numbers β such that the greedy and lazy expansions of 1 are equal. In [20], a combinatorial method of determining when an expansion is greedy or unique is given. We call a number β *univoque* if $d_\beta(1) = \ell_\beta(1)$ (or equivalently, if 1 has a unique expansion base β). The set of all such numbers is defined by $\mathcal{U}$. Of the set of univoque numbers, $\mathcal{U}$, there is a smallest such number [29], $\kappa \approx 1.787231\cdots$. Moreover, κ is transcendental [1] and is not isolated [32]. In [31], Komornik and Loreti showed that $\overline{\mathcal{U}}$ is a perfect set, and moreover, that since $\mathcal{U}$ has measure 0, then $\mathcal{U}$ is a Cantor set. In the case when the expansions are periodic or finite, this method can be made algorithmic [2]. In particular, in [2] the authors look at Pisot numbers with respect to these expansions. This is explained in more detail in Section 2.

When the greedy and lazy expansions are not equal, occasionally there are an infinite number of different general beta-expansions. In [18] and [19] it is shown how to create "quasi-greedy" expansions, which will fall in between the greedy and the lazy expansions.

Another, even more general idea is given in [30], where they generalized the idea of expansions to $\sum c_i \cdot p_i$ where $c_i \in \mathbb{Z}$, $0 \leq c_i \leq m_i$ and $p_i \to 0$. This is the same as the standard beta-expansion when $p_i = \beta^{-i}$ and $m_i = \lceil \beta \rceil$. Another interesting variation on the idea of beta-expansions includes the situation where the base β is a complex number. For example, see [34, Chapter 7], and also [5, 28].

2. Univoque Pisot numbers

The goal of the talk at WWCA was to discuss some recent work of the author with Allouche and Frougny concerning their investigation of univoque Pisot numbers. Recall a Pisot number is a real root greater than 1 of a monic integer polynomial, such that all of its conjugates have modulus strictly less than 1. A number β is univoque if the greedy and lazy expansions of 1 are the same. The main questions of the investigation were:

- Are there any univoque Pisot numbers?
- Is there a smallest univoque Pisot number?
- Are there any infinite families of univoque Pisot numbers?

Limit Points	Defining polynomials
φ_r	$\Phi_r(x)x^n \pm (x^r - x^{r-1} + 1)$
	$\Phi_r(x)x^n \pm (x^r - x + 1)$
	$\Phi_r(x)x^n \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi_r(x)x^n \pm (x^{r+1} - 1)$
	$\Psi_r(x)x^n \pm (x^r - 1)/(x - 1)$
χ	$\mathcal{X}(x)x^n \pm (x^3 + x^2 - x - 1)$
	$\mathcal{X}(x)x^n \pm (x^4 - x^2 + 1)$

Table 2: Regular Pisot numbers

- What sort of structure does the set of univoque Pisot numbers have?

The answer to the first three questions were all yes. Before discussing the last question (and how we arrived at the first three answers) we need to discuss the structure of the set of Pisot numbers in more detail.

We denote the set of Pisot numbers by S. Amara has determined all the limit points of S smaller than 2 in [3].

Theorem 2.1. [3] *The limit points of S in $(1, 2)$ are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < \psi_r < \varphi_{r+1} < \cdots < 2$$

where

$$\begin{cases} \text{the minimal polynomial of } \varphi_r \text{ is } \Phi_r(x) = x^{r+1} - 2x^r + x - 1, \\ \text{the minimal polynomial of } \psi_r \text{ is } \Psi_r(x) = x^{r+1} - x^r - \cdots - x - 1, \\ \text{the minimal polynomial of } \chi \text{ is } \mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1. \end{cases}$$

The first few limit points are:

- $\varphi_1 = \psi_1 \approx 1.618033989$, the root in $(1, 2)$ of $\Psi_1(x) = \Phi_1(x) = x^2 - x - 1$
- $\varphi_2 \approx 1.754877666$, the root in $(1, 2)$ of $\Psi_2(x) = x^3 - 2x^2 + x - 1$
- $\psi_2 \approx 1.839286755$, the root in $(1, 2)$ of $\Psi_2(x) = x^3 - x^2 - x - 1$
- $\varphi_3 \approx 1.866760399$, the root in $(1, 2)$ of $\Phi_3(x) = x^4 - 2x^3 + x - 1$
- $\chi \approx 1.905166168$, the root in $(1, 2)$ of $\mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1$
- $\psi_3 \approx 1.927561975$, the root in $(1, 2)$ of $\Psi_3(x) = x^4 - x^3 - x^2 - x - 1$

Not only did Amara completely determine the limit points, he also gave a description of the regular Pisot numbers approaching these limit points. For each of these limit points (φ_r, ψ_r or χ), there exists an ϵ, (dependent on the limit point) such that all Pisot numbers in an ϵ-neighbourhood of this limit point are these regular Pisot numbers. The Pisot root of the defining polynomial approaches the limit point as n tends to infinity. The defining polynomials for these regular Pisot numbers are given in Table 2. It should be noted that these polynomials are not necessarily minimal, and may contain some cyclotomic factors. Also, they are only guaranteed to have a Pisot number root for sufficiently large n (although for our purposes, $n = 1$ or 2 normally is sufficiently large).

The key observation used for this study was that "nice" sequences of regular Pisot numbers approaching a limit point give "nice" sequences of greedy and lazy expansions. This meant that it was possible to determine infinite classes of Pisot numbers as being univoque or not univoque. Along with the observation that χ was a univoque Pisot number, our new goal became:

Minimal Polynomial	Pisot Number	Greedy expansion	Lazy expansion	Comment
$x^{r+1} - 2x^r + x - 1$	φ_r	$1^r0^{r-1}1$	$1^{r-1}01^\omega$	
$x^{r+1} - x^r - \cdots - 1$	ψ_r	1^{r+1}	$(1^r0)^\omega$	
$x^4 - x^3 - 2x^2 + 1$	χ	$11(10)^\omega$	$11(10)^\omega$	Univoque

Table 3: Greedy and lazy β-expansions of real numbers in $S' \cap (1,2)$.

Case	Greedy expansion	Lazy expansion	Comment
	$\Psi_2(x)x^n - (x+1)$		
$n = 1$	Root bigger than 2		
$n = 4$	1110011	$(1110010)^\omega$	
$n = 3k+1$	$11100(000)^{k-1}11$	$(11(011)^{k-1}1001(101)^{k-1}0)^\omega$	
	$\mathcal{X}(x)x^n - (x^3 + x^2 - x - 1)$		
$n = 2$	Root bigger than 2		
$n = 2k+2$	$111(01)^{k-1}1011((10)^{k-1}0111(01)^{k-1}1000)^\omega$	$111(01)^{k-1}1011((10)^{k-1}0111(01)^{k-1}1000)^\omega$	Univoque

Table 4: Greedy and lazy expansions for some regular Pisot numbers.

- Find the greedy and lazy expansions of all regular Pisot numbers approaching limit points less than or equal to χ.
- Find all Pisot numbers less than χ by Boyd's algorithm, removing from the search those regular Pisot numbers accounted for in the previous step. After this, determine which of these Pisot numbers are univoque.

Some examples of the types of patterns found are given in Table 3 and 4. For a more complete list, see [2]. Table 3 shows us that χ is a univoque Pisot number. Moreover, we see from Table 4 that χ is the limit point of univoque Pisot numbers. Using a similar analysis, it was shown that there were no other regular Pisot numbers approaching a limit point less that χ that was univoque. Using Boyd's method [12, 13], all relevant Pisot numbers, not accounted for above, less than χ were enumerated and tested to see if they were univoque. (Some intervals of Pisot numbers could be eliminated based on combinatorial arguments that aren't of interest to this survey.) This gives us the result that there are exactly two univoque Pisot numbers less than χ. They are

- $1.880000\cdots$ the root in $(1,2)$ of the polynomial $x^{14} - 2x^{13} + x^{11} - x^{10} - x^7 + x^6 - x^4 + x^3 - x + 1$ with univoque expansion $111001011(1001010)^\omega$.
- $1.886681\cdots$ the root in $(1,2)$ of the polynomial $x^{12} - 2x^{11} + x^{10} - 2x^9 + x^8 - x^3 + x^2 - x + 1$ with univoque expansion $111001101(1100)^\omega$

3. Algorithms and Implementation Issues

One of the main computations needed for computing the greedy expansion is the calculation of $a_n = \lfloor \beta \cdot r_{n-1} \rfloor$. This must be done as a floating point calculation, as Maple is unable to determine which integer this should be symbolically. Unfortunately the introduction of floating point numbers allows for the introduction of rounding error. To protect against rounding error, we test $|a_n - \beta \cdot r_{n-1}|$ to a tolerance tol. For our purposes, we use tol as $10^{-Digits/2}$ where $Digits$ is the number of digits of accuracy we are computing to. If we have that $|a_n - \beta \cdot r_n| < tol$ then we assume that we are not working with a high enough accuracy, and we double $Digits$, recompute β to this higher degree of accuracy, and then redo the calculation.

In contrast to this, the calculation for r_n is always done symbolically. This is done for two reasons. The first is that it prevents the accumulation of round-off error through successive calculations. The second reason is that it allows for the detection of the beta-expansion begin periodic or finite, as we compare r_n to all r_k with $k < n$. Similar issues needed to be addressed for computing the lazy expansion. An example of the code being run is given below. The code itself can be found at [24].

Example 3. *The code below assumes that β is the unique root of the polynomial between 1 and 2. Also, as Pisot numbers are known to be beta-numbers, they are always completely determined. When the polynomial does not have a Pisot root, then there is no guarantee that the root is a beta-number, so the greedy or lazy expansion may not have a nice closed form. For this reason, only the first 60 digits are computed. If in the first 60 digits, the expansion is determined to be periodic, then this information is presented, otherwise, only the first 60 digits are presented. By default, the expansion of 1 is given.*

```
> read all;
> PPGreedy(x^3-x-1);
                              "10001(0)^omega"

> PPLazy(x^3-x-1);
                               "0000(1)^omega"

> PPGreedy(x^4-x-1);
"Warning: Polynomial is not Pisot.  Only Computing first 60 terms"
       "100000001000000000000100000000100000000001000000000000000001..."

> PPLazy(x^4-x-1);
"Warning: Polynomial is not Pisot.  Only Computing first 60 terms"
       "000000011101111111111110111111110111111111101111111111111111..."
```

One interesting observation made, which allows for the classification of regular Pisot numbers is that "nice" sequences of regular Pisot numbers give "nice" sequences of beta-expansions.

Example 4.

```
> read all;
> for n from 1 to 5 do n, `PPGreedy`(x^(n+1)-2*x^n+x-1); od;
                              1, "11(0)^omega"

                             2, "1101(0)^omega"

                            3, "111001(0)^omega"

                           4, "11110001(0)^omega"
```

```
                         5, "1111100001(0)^omega"

> for n from 1 to 5 do n, `PPLazy`(x^(n+1)-2*x^n+x-1); od;
                              1, "0(1)^omega"

                              2, "10(1)^omega"

                              3, "110(1)^omega"

                              4, "1110(1)^omega"

                              5, "11110(1)^omega"
```

Now some sequences of expansions are "nicer" than others, and it would be time consuming and prone to errors to find all of these patterns by hand. For that reason, algorithms were developed that:

- Predicted the sequences of expansions based upon the first few terms in the sequence.
- Based upon this prediction, predicted the sequence of companions polynomials.
- Based upon these companion polynomials, showed that the defining polynomial divides the companion polynomial, thus showing that the predicted sequences gives a valid sequence of expansions. This actually required the prediction of the co-factor polynomials.
- Use combinatorial properties (not discussed here) to show that these predicted sequences of expansions are greedy or lazy expansions as necessary, (see [2] for more details).

Example 5. *In the next example, we first have a strange definition of P, as every other defining polynomial has a cyclotomic factor of $x+1$. When we list six terms, no patterns becomes apparent for the lazy expansion. But if we look at the every third expansion, a pattern does present itself, and it is easy to prove this pattern.*

```
> read all;
> P := (n,x)->`if`(type(n, odd),
         (x^3-x^2-x-1)*x^n-x^3+1,
         simplify(((x^3-x^2-x-1)*x^n-x^3+1)/(x+1))):

> for n from 3 to 8 do n, PPGreedy(P(n,x)), PPLazy(P(n,x)); od;
                    3, "111(110)^omega", "111(110)^omega"

                  4, "111(0110)^omega", "111(0110)^omega"

             5, "111(00110)^omega", "1110010111(11110)^omega"

                6, "111(000110)^omega", "110111(110)^omega"

             7, "111(0000110)^omega", "110111(0110110)^omega"
```

```
        8, "111(00000110)^omega", "1101110011010111(11011110)^omega"

> for n from 3 to 18 by 3  do n, PPGreedy(P(n,x)), PPLazy(P(n,x)); od;
                   3, "111(110)^omega", "111(110)^omega"

                6, "111(000110)^omega", "110111(110)^omega"

             9, "111(000000110)^omega", "110110111(110)^omega"

         12, "111(000000000110)^omega", "110110110111(110)^omega"

      15, "111(000000000000110)^omega", "110110110110111(110)^omega"

   18, "111(000000000000000110)^omega", "110110110110110111(110)^omega"

> CompleteConj2(n->P(n, x), 6, 3, k, x);
Looking at n = 3 k + 3
Greedy Expansion: 111((000)^(k) 110)^omega
- Univogue Exceptions: k = 0 is univoque
- Greedy Expansion valid for k >= 0
Lazy Expansion: 11(011)^(k) 1(110)^omega
- Lazy Expansion is valid for k >= 0
```

The way the output from CompleteConj2 should be interpreted is

- *Let* β_n *be the Pisot root of* $(x^3 - x^2 - x - 1)x^n - x^3 + 1$.
- *Then the greedy expansion for* β_{3k+3} *is* $111((000)^k 110)^\omega$.
- *The lazy expansion for* β_{3k+3} *is* $11(011)^k 1(110)^\omega$.
- *Both of these expansions are valid for all* k.
- *In the case* $k = 0$ *this expansion is univoque.*

4. Conclusions and Open Questions

For convince we restate the three conjectures that were given in this paper.

Conjecture (Schmidt's Conjecture). *We have* $\mathbb{Q} \cap [0,1) \subset \mathrm{Per}(\beta)$ *if and only if* β *is a Pisot or a Salem number.*

Conjecture (Boyd's Conjecture). *The greedy expansion* $d_\beta(1)$ *is eventually periodic for Salem numbers* β *of degree 6, but the expansion is not eventually periodic for a positive proportion of Salem numbers of higher degrees.*

Conjecture (Lehmer's Conjecture). *The smallest Salem number is* $1.1762\cdots$, *the root of* $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

Some other interesting questions worth investigating are:

- In general, are the greedy/lazy β-expansions periodic for Salem numbers? (This is not known to be true, see [16] for more details.)
- It is known that Pisot numbers can be written as a limit of Salem numbers, where if $P(x)$ is the minimal polynomial of a Pisot number, then $P(x)x^n \pm P^*(x)$ has a Salem number as a root, which tends to the root of the Pisot number. Some preliminary and somewhat haphazard investigation suggests that we might be able to find a "nice" looking expression

for the greedy (resp. lazy) beta-expansion of these Salem numbers, which tends towards the greedy (resp. lazy) beta-expansion of the Pisot number. If true, then this could have implications towards questions concerning the beta-expansions of Salem numbers being eventually periodic.

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