ZEROS OF POLYNOMIALS WITH CYCLOTOMIC COEFFICIENTS

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Abstract. This paper examines properties of the zeros of polynomials with restricted coefficients. In particular we study the case when the coefficients are restricted to the roots of unity and possibly zero. The methods used in this paper are adaptations of methods used by Odlyzko and Poonen in "*Zeros of Polynomials with 0, 1 Coefficients*". The main result of this paper is that the closure of the set of zeros of polynomials with n^{th} roots of unity as coefficients or with n^{th} roots of unity and 0 as coefficients, are path connected for all integers $n \geq 1$.

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1. Introduction

Let $p(z) = a_n z^n + \cdots + a_0$ be a polynomial. If the coefficients a_i are restricted in some way, then there is interest in how this restriction affects the roots, or multiple roots, of $p(z)$. For instance [2] studies how small a root of multiplicity k can be for a polynomial with coefficients in $\{1, -1\}$. Another example is [6] where it is shown that a polynomial

$$p(x) = \sum_{j=0}^{n} a_j x^j, \ |a_0| = 1, \ |a_j| \leq 1$$

has at most $c\sqrt{n}$ zeros inside a polygon with vertices on the unit circle, and $\frac{c}{\alpha}$ zeros in a polygon with vertices on $\{z \in \mathbb{C} : |z| = 1 - \alpha\}$ where $\alpha \in (0, 1)$ and c depends on the number of vertices.

In addition, sometimes the point of interest is the root-free region of restricted polynomials, such as in [1]. Or how restricted coefficients

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affect properties of the polynomials themselves. Such as in [9] where it is shown that the minimal length of a polynomial with coefficients in $\{1, -1\}$ with a zero of given order m at $x = 1$ (denoted $N^*(m)$) is given by:

$$N^*(m) = 2^m,\ \forall m \leq 5,\ N^*(6) = 48,\ N^*(7) = 96.$$

One problem of interest is, given a root α and set S, decide if there exists a polynomial $p(z)$ with all of its coefficients in S such that α is a root of $p(z)$, (possibly to a desired multiplicity). A number of attacks upon this problem have involved Siegel's Lemma, as described in [4, 5]. These approaches tend to be non-constructive, proving existence without giving an example.

More constructive methods are described by Borwein and Mossinghoff, [7], where they looked at polynomials where the coefficients were either 0 or 1. In particular, they studied how often -1 could be a factor of a polynomial of degree n of this form. More generally Mossinghoff, [16], considered the case of non-cyclotomic roots of polynomials with restricted coefficients. He answered an open question of Odlyzko and Poonen's, finding a double root of $\{0, 1\}$ polynomials other than a cyclotomic root. He further looked at the problem of finding multiple roots of $\{-1, 1\}$ and $\{-1, 0, 1\}$ polynomials, as well as considering their Mahler measure.

As opposed to considering the question of where an algebraic number could be found exactly as a root of a polynomial with restricted coefficients, Borwein and Pinner, [8], considered how well it could be approximated. Given an algebraic number α, they studied how well it could be approximated by a polynomial with coefficients $\{0, +1, -1\}$. They found that the worst approximations tended to be near the roots of unity. They have some very nice pictures of the fractal like set which is made up of the roots of polynomials with $\{0, +1, -1\}$ coefficients.

The study of when the coefficients are taken randomly from a distribution as opposed to being taken from a finite set, is done in [3]. Pictures similar to those found in [8] can also be found in this paper.

A number of bounds have been found for the roots of polynomials with $\{0, 1\}$ coefficients. It was shown in [10] that if z is a root of a $\{0, 1\}$ polynomial then $\Re(z) \leq 1.4655...$, the roots of $x^3 - x^2 - 1$. (Odlyzko and Poonen simply approximated this to 3/2.) In fact [10] showed a more general result, discussing more general polynomials than those with just $\{0, 1\}$ coefficients. Independently Flatto, Lagarias, and Poonen, as well as Solomyak showed for z a root of a $\{0, 1\}$ polynomial, that $1/\phi \leq |z| \leq \phi$, where ϕ is the golden ratio [13, 18].

A more restrictive bound was given by [17], which showed that the roots were between the two curves

$$C_1 = \left\{ z : |z| \leq 1, \frac{|z|}{1-|z|} = \left| \frac{2-z}{1-z} \right| \right\}$$

and

$$C_2 = \left\{ z : |z| \leq 1, \frac{|z|}{|z|-1} = \left| \frac{2-z}{1-z} \right| \right\}.$$

They also showed that the line segment $[-\phi, -1/\phi]$ is in this set.

Of interest to Poonen and Odlyzko was whether the closure of the set of roots was path connected or not. This set exhibits a fractal like appearance, and something can be said about the connectivity of a number of these types of sets. For example, the Mandelbrot set is connected. See [14] for a complete list of references. Most importantly for this paper, Odlyzko and Poonen showed that the closure of the set of roots of $0, 1$ polynomials was indeed path connected.

This paper discusses other possible sets S, such that if the $a_i \in S$, then the closure of the roots is path connected. The sets of coefficients which are considered are the n^{th} roots of unity alone and also this set plus 0.

Definition 1.1. *Let $n \in \mathbb{N}$. We denote by C_n, the n^{th} roots of unity. Formally*

$$C_n = \left\{ \exp\left(\frac{2\pi i k}{n}\right) : k \in \mathbb{Z} \right\}.$$

Definition 1.2. *Let $A \subset \mathbb{C}$. We denote by Ω_A the set of (non-trivial) roots of polynomials with coefficients in A. We assume that $a_0 \neq 0$, to avoid the trivial root of 0. Formally:*

$$\Omega_A = \{z : a_0 + a_1 z + \cdots + a_n z^n = 0, n \in \mathbb{N}, a_i \in A, a_0 \neq 0\}.$$

For convenience, we denote $\Omega_n := \Omega_{C_n}$ and $\Omega_{n,0} := \Omega_{C_n \cup \{0\}}$.

In Section 1 we display several properties of Ω_n and $\Omega_{n,0}$. This includes its symmetries, location, and an upper bound on the number of a zeros of a particular polynomial within the unit circle. In Section 2 we prove that there exists a neighborhood of the unit circle in the complex plane, except at real values ± 1, which lies inside $\overline{\Omega_n} \cap \overline{\Omega_{n,0}}$. Finally in Section 3 we prove the main theorem of the paper, along with two necessary lemmas.

Much of the work in this paper was heavily influenced by Andrew M. Odlyzko and Bjorn Poonen's paper "*Zeros of polynomials with 0,1 coefficients*" [17].

2. Locations and Bounds on Zeros

We start this section with a classic result of Cauchy, (see for instance Theorem 3.1.2, page 244, of [15].)

Theorem 2.1 (Cauchy, 1829). *If $a_i \in \mathbb{C}$ and $a_0 a_n \neq 0$, all complex roots of $a_0 + \ldots + a_n z^n$ lie in the annulus $\frac{1}{M_1+1} < |z| < M_2 + 1$ where $M_1 = \max_{i\neq 0} |\frac{a_i}{a_0}|$, $M_2 = \max_{i\neq n} |\frac{a_i}{a_n}|$.*

Corollary 2.1. *If $z \in \Omega_n$ or $z \in \Omega_{n,0}$ then $\frac{1}{2} < |z| < 2$.*

Proposition 2.1. *Let $\Omega := \Omega_n$ or $\Omega_{n,0}$. If $z \in \Omega$ then $\overline{z}, \frac{1}{z} \in \Omega$. Furthermore, if n is even, $-z \in \Omega$.*

Proof. Let α be a root of $a_0 + a_1 z^1 + \ldots + a_n z^n$, $a_i \in \mathbb{C}$. Then $\overline{\alpha}$, the complex conjugate of α, is a root of $\overline{a_0} + \overline{a_1} z + \ldots + \overline{a_n} z^n$, and $\frac{1}{\alpha}$ is a root of $a_n + a_{n-1} z + \ldots + a_0 z^n$. These have coefficients in the desired set. Further $-\alpha$ is a root of $a_0 - a_1 z^1 + \ldots + (-1)^n a_n z^n$, and if n is even, this also has coefficients from the desired set. □

Proposition 2.2. *Suppose $p(z)$ is a power series of the form*

$$p(z) = \sum_{k=0}^{\infty} a_i z^k$$

where $0 \neq |a_0| \geq 1$ and $a_i \in A$, where A is a finite subset of $\mathbb{C}$. Then for any disk of radius r, $0 < r < 1$, $p(z)$ has $\leq -2(\log r)^{-1} \left(\log \frac{a}{1-r^{1/2}}\right)$ zeros on or inside said disk, where $a = \max_{a_i \in A} |a_i|$.

Proof. Consider a disk of radius R, $r < R < 1$. Say that $z_1, \ldots, z_n$ are the zeros in the disk of radius R. By Jensen's Theorem (Theorem 3.61 of [19]), we can say

$$\log \frac{R^n |p(0)|}{|z_1| \cdots |z_n|} = \log \frac{R^n |a_0|}{|z_1| \cdots |z_n|} = \frac{1}{2\pi} \int_0^{2\pi} \log |p(Re^{i\theta})| d\theta.$$

Suppose m is the number of zeros in $|z| < r$, so $m \leq n$. Then

$$\log \frac{R^n |a_0|}{|z_1| \cdots |z_n|} = \log \frac{R^m}{|z_1| \cdots |z_m|} + \log \frac{R^{n-m} |a_0|}{|z_{n-m}| \cdots |z_n|}.$$

Since $\log \frac{R^{n-m}|a_0|}{|z_{n-m}| \cdots |z_n|} \geq 0$ then we have

$$\log \frac{R^m}{|z_1| \cdots |z_m|} \leq \frac{1}{2\pi} \int_0^{2\pi} \log |p(Re^{i\theta})| d\theta.$$

Now since $\log r^m \geq \log(|z_1|\cdots|z_m|)$ we get

$$\log R^m - \log r^m \leq \log R^m - \log(|z_1|\cdots|z_m|) \leq \frac{1}{2\pi}\int_0^{2\pi} \log|p(Re^{i\theta})|d\theta.$$

Now if we evaluate $p(Re^{i\theta})$ we get

$$\left|p\left(Re^{i\theta}\right)\right| = \left|\sum_{k=0}^{\infty} a_i \left(Re^{i\theta}\right)^k\right| \leq \sum_{k=0}^{\infty} |a_i| R^k \leq a \sum_{k=0}^{\infty} R^k = \frac{a}{1-R}.$$

Thus $m \leq (\log R - \log r)^{-1}(\log \frac{a}{1-R})$. If we set $R = r^{1/2}$ we get our bound,

$$m \leq -2(\log r)^{-1}\left(\log \frac{a}{1-r^{1/2}}\right).$$

□

Corollary 2.2. *Suppose $p(z)$ is a power series of the form*

$$p(z) = \sum_{k=0}^{\infty} a_i z^k$$

where $a_0 \neq 0$ and either $a_i \in C_n$ or $a_i \in C_n \cup \{0\}$. Then for any disk of radius r, $0 < r < 1$, $p(z)$ has $\leq -2(\log r)^{-1}(\log \frac{1}{1-r^{1/2}})$ zeros on or inside this disk.

Proof. This follows immediately from Proposition 2.2. □

3. Roots near the unit circle

The goal of this section is to show that a neighborhood of the unit circle (exception possibly ± 1) is in the interior of $\overline{\Omega_n}$ and $\overline{\Omega_{n,0}}$.

Lemma 3.1. *If $B \subseteq \mathbb{C}$ is compact, $k \geq 1$, $|z| < 1$, and*

$$B \subseteq \bigcup_{a_1, a_2, \ldots, a_k \in A} \left[\left(\sum_{i=1}^{k} a_i z^i\right) + z^k B\right]$$

then every element of B is of the form

$$\sum_{i=1}^{\infty} a_i z^i, \; a_i \in A \subset \mathbb{C}.$$

In particular, if $-1 \in B$ and $1 \in A$, then $z \in \overline{\Omega_A}$.

Proof. Given $b_m \in B$, inductively pick $b_{m+1} \in B$ and $a_{m,i} \in A,\ m \geq 0$, $1 \leq i \leq k$ such that

$$b_m = \left(\sum_{i=1}^{k} a_{m,i} z^i\right) + z^k b_{m+1} \ .$$

Successive substitution yields

$$b_0 = \left(\sum_{m=0}^{M-1} \sum_{i=1}^{k} a_{m,i} z^{mk+i}\right) + z^{Mk} b_M \ .$$

Since B is compact and $|z| < 1$ we have $z^{Mk} b_M \to 0$ as $M \to \infty$, so

$$b_0 = \sum_{m=0}^{\infty} \sum_{i=1}^{k} a_{m,i} z^{mk+i}.$$

Thus all $b_m = \sum_{i=1}^{\infty} a_i z^i$, with $a_i \in A$. So consider $-1 \in B$ and $1 \in A$. Then $-1 = \sum_{i=1}^{\infty} a_i z^i$ implies $0 = 1 + \sum_{i=1}^{\infty} a_i z^i$, so $z \in \overline{\Omega_A}$. □

Lemma 3.2. *If* $B \subseteq \mathbb{C}$ *is compact,* $-1 \in B,\ k \geq 1,\ z_0 \in \mathbb{C},\ 1 \in A$ *and*

$$B \subseteq \operatorname{int} \bigcup_{a_1,\ldots,a_k \in A} \left[\left(\sum_{i=1}^{k} a_i z_0^i\right) + z_0^k B\right],$$

where int S *denotes the interior of* S*, then there is a neighborhood* N *of* z_0 *such that* $(N \cap \{z : |z| < 1\}) \subseteq \overline{\Omega_A}$.

Proof.

$$B \subseteq \operatorname{int} \bigcup_{a_1,\ldots,a_k \in A} \left[\left(\sum_{i=1}^{k} a_i z_0^i\right) + z_0^k B\right] \ ,$$

implies that

$$B \subseteq \bigcup_{a_1,a_2,\ldots,a_k \in A} \left[\left(\sum_{i=1}^{k} a_i z^i\right) + z^k B\right]$$

holds for z in a neighborhood of z_0, so Lemma 3.2 follows from Lemma 3.1. □

Lemma 3.3. $\{z : \ |z| = 1,\ z \in \mathbb{C}\} \subset \left(\overline{\Omega_n} \cap \overline{\Omega_{n,0}}\right)$.

Proof. If z_0 is a root of unity, then z_0 is a zero of

$$p(z) = \sum_{k=0}^{m-1} z^k,$$

for some m. Thus $\{e^{\frac{2\pi i k}{m}} \ : \ k, m \in \mathbb{Z}\} \subset (\Omega_n \cap \Omega_{n,0})$. Thus the closure of this set, the unit circle, is contained in $\left(\overline{\Omega_n} \cap \overline{\Omega_{n,0}}\right)$. □

Proposition 3.1. *If* $|z| = 1$, $z \neq \pm 1$ *then* $z \in \operatorname{int}\left(\overline{\Omega_n} \cap \overline{\Omega_{n,0}}\right)$.

Proof. We see from [17] that there exists a neighborhood N with the desired property for $\Omega_{1,0}$. Further we have that $N \subset \Omega_{1,0} \subset \Omega_{n,0}$ for all n, so it suffices to prove the result for Ω_n.

If $n \geq 3$ and $B = \{z : |z| \leq 2\}$ we have that

$$B \subset \operatorname{int} \bigcup_{a_i \in C_n} (a_i z + B)$$

for all $|z| = 1$, by considering this as a problem in geometry. Hence, this follows from Lemma 3.3.

For $n = 2$, we need to consider

$$\bigcup (\pm z \pm z^2 + B)$$

with $|z| = 1$ and $z \neq \pm 1$. Here again, we have that B is in the interior, as can be seen by determining the intersection points of the circles $|\zeta \pm z \pm z^2| = 2$, and showing that the exterior ones are always greater than 2 in modulus. So this again follows from Lemma 3.3. □

4. Main Result

Give $A = \{a_1, a_2, \cdots, a_k\}$ and A^ω the usual discrete and product topologies respectively, where $a_i \in \mathbb{C}$. We use the following notation; $v = (v_1, v_2, \cdots, v_m)$ is a finite vector where $v_i \in A$, and S_v is the set of sequences in A^ω which start with v. Also va_i is the vector v with a_i appended.

Lemma 4.1. *Let* $A = \{a_1, a_2, \cdots, a_k\}$ *be a finite set of complex numbers. Let* T *be a topological space. Suppose* $f : A^\omega \to T$ *is a continuous map such that*

$$f(S_{va_i}) \cap f(S_{va_j}) \neq \emptyset \tag{1}$$

for all $a_i, a_j \in A$, *and for all* $v \in A^m$, $m \geq 0$. *Then the image of* f *is path connected.*

Proof. The idea of the proof is to start with two points, $w(0) = f(x'_0)$ and $w(1) = f(x_1)$, in $image(f)$ and to connect them by a path by inductively "filling in" a continuum of points between them. Begin by finding $x_{1/2}, x'_{1/2} \in A^\omega$ such that x'_0 and $x_{1/2}$ (respectively $x'_{1/2}$ and x_1) agree in the first coordinate and $f(x_{1/2}) = f(x'_{1/2})$. Such points are guaranteed by (1). That is, if x'_0 and x_1 have the same first coordinate, let $x'_0 = x_{1/2} = x'_{1/2}$ otherwise apply (1) with v as the null vector. We then say $f(x_{1/2}) = f(x'_{1/2}) = w(\frac{1}{2})$. Now using the same procedure we find $x_{1/4}, x'_{1/4} \in A^\omega$ such that x'_0 and $x_{1/4}$ agree in the first two coordinates and $f(x_{1/4}) = f(x'_{1/4})$ which we define as $w(\frac{1}{4})$. We do the analogous for $w(\frac{3}{4})$. Notice that if we continue this induction we can find for all dyadic rationals $\frac{d}{2^m} \in [0,1]$, $x_{\frac{d}{2^m}}, x'_{\frac{d}{2^m}}$ and $w(\frac{d}{2_m})$ such that $x'_{\frac{d}{2^m}}$ and $x_{\frac{d+1}{2^m}}$ agree in the first m-coordinates and

$$w\left(\frac{d}{2^m}\right) = f\left(x'_{\frac{d}{2^m}}\right) = f\left(x_{\frac{d}{2^m}}\right).$$

Also we see that for $q \in \left[\frac{d}{2^m}, \frac{d+1}{2^m}\right)$ all x'_q agree in the first m coordinates.

So now we need to define something similar for non-dyadic rationals. For r not a dyadic rational we define

$$r = \sum_{i=1}^{\infty} \frac{e_i}{2^i} \in [0,1]$$

and $w(r) = \lim_{m\to\infty} w\left(\sum_{i=1}^{m} \frac{e_i}{2^i}\right) = f(x_r)$. The idea here is that all non-dyadic rationals in $[0,1]$ can be expressed as an infinite binary expansion. So given this definition ω maps $\left[\frac{d}{2^m}, \frac{d+1}{2^m}\right]$ into $f(S_v)$ where $v \in A^m$ is the first m coordinates of $x'_r, r \in \left[\frac{d}{2^m}, \frac{d+1}{2^m}\right)$ and of $x_{\frac{d+1}{2^m}}$. So now consider an open subset U of T, containing $w(r)$ where $r \in [0,1]$. Then because f is continuous

$$(S_v) \cup (S_{v'}) \subset f^{-1}(U)$$

for some v and v' of x_r and x'_r respectively. So

$$f(S_v) \cup f(S_{v'}) \subset U$$
$$\Rightarrow w^{-1}(f(S_v) \cup f(S_{v'})) \subset w^{-1}(U)$$

will contain a neighborhood of r because $w : \left[\frac{d}{2^m}, \frac{d+1}{2^m}\right] \to f(S_v)$. Thus it follows that $w : [0,1] \to image(f)$ is a continuous path and hence $image(f)$ is path connected. □

The next lemma is Lemma 5.1 from [17].

Lemma 4.2. (Lifting lemma): *Let M be a Hausdorff space and let $\pi : M^n \to M^n/S_n$ be the projection map. Let $f : [0,1] \to M^n/S_n$ be a continuous map. Then there is a continuous map $g : [0,1] \to M^n$ such that $f = \pi \circ g$.*

This now gives the main result.

Theorem 4.1. *$\overline{\Omega_n}$ and $\overline{\Omega_{n,0}}$ are path connected.*

Proof. Notice that $\overline{\Omega_{1,0}}$ is proved to be path connected by Odlyzko and Poonen. Further $\overline{\Omega_1}$ is simply the unit circle. So it suffices to consider the case when $n \geq 2$. Let A be equal to either C_n or $C_n \cup \{0\}$. Let U

be the unit disk $\{z : |z| \le 1\}$, with the unit circle shrunk to a point, P. So we can imagine U, from a topological point of view, as a sphere. Thus we can give it a bounded metric d. We wish to use our metric to prove continuity so we need a metric which is always finite, non-zero, and also minimal. Consider U^∞ as the set of sequences $x = \{x_i\}_{i=1}^\infty$ which converge to P and define a metric d_∞ on U^∞ by

$$d_\infty(x, y) = \sup_i d(x_i, y_i).$$

Let S_∞ be the group of permutations of $\{1, 2, \ldots\}$ which acts on U^∞ by permuting the coordinates. We are interested in $\frac{U^\infty}{S_\infty}$ and its metric D which we define as

$$D(\overline{x}, \overline{y}) = \inf_{\sigma \in S_\infty} d_\infty(x, \sigma y),$$

where $\overline{x}$ denotes the projection of $x \in U^\infty$ into $\frac{U^\infty}{S_\infty}$. For a closed disk inside the unit circle we showed that a power series of the form

$$1 + a_1 z + a_2 z^2 + \ldots, \ a_i \in A \subset \mathbb{C}$$

has a bounded number of zeros (Corollary 2.2). We may assume without loss of generality that $a_0 = 1$, because of the structure of A. Thus for the disk $\{z : |z| < 1\}$ the zeros of the power series are either finite or form a sequence converging to P. If a power series yields a finite number of zeros then we attach an infinite sequence of P's. We can define the map

$$f : A^\omega \to \frac{U^\infty}{S_\infty}.$$

To show continuity of this map we apply Roche's theorem. If two power series agree in the first m coordinates for m sufficiently large then their zeros inside $\{z : |z| < 1\}$ will be within ϵ because of the topology we have defined.

Now we need to show that (1) holds from Lemma 4.1. To do this consider $a_k \in A$, and define

$$w_k = (v_1, \ldots, v_m, a_k, a_k v_1, \ldots, a_k v_m, a_k^2, a_k^2 v_1, \ldots).$$

So $w_k \in S_{va_k}$. Now we see that

$$\begin{aligned} & 1 + v_1 z + \ldots + v_m z^m + a_k z^{m+1} + a_k v_1 z^{m+2} + \ldots \\ & a_k v_m z^{2m+1} + a_k^2 z^{2m+2} + a_k^2 v_1 z^{2m+3} + \ldots \\ = \;& (1 + v_1 z + \ldots + v_m z^m)\left(1 + a_k z^{m+1} + a_k^2 z^{2(m+1)} + \ldots\right) \\ = \;& (1 + v_1 z + \ldots + v_m z^m)\frac{1}{1 - a_k z^{m+1}} \end{aligned}$$

Thus for any $a_i, a_j \in A,\ a_i \neq a_j$ which are coefficients in the above power series, we have the same zeros inside $\{z : |z| < 1\}$. Thus $f(w_i) = f(w_j)$ and hence condition (1) holds. Thus we can apply Lemma 4.1 to deduce that $image(f)$ is path connected.

Suppose $z_0 \in \left(\overline{\Omega_A} \cap \{z : |z| < 1\}\right)$. Let $\beta : [0,1] \to \frac{U^\infty}{S_\infty}$ be a path from $f((\gamma))$ to $f((1,1,\ldots)) = \{P, P, \ldots\}$ where $\gamma = (1, a_1, a_2, \ldots)$ with $1 + \sum_{k=1}^{\infty} a_i z_0^k = 0$ and $a_i \in A$. Let U_λ be $\{z : |z| \leq 1\}$ with annulus $\{z : 1 - \frac{1}{\lambda} \leq |z| \leq 1\}$ shrunk to a point Q for $\lambda \geq 1$. Define $|\cdot|$ on U_λ as $|Q| = 1 - \frac{1}{\lambda}$. Again by Corollary 2.2 we know there is an upper bound on the number of zeros of power series inside $\{z : |z| < 1 - \frac{1}{\lambda}\}$, say m. What we do is apply the projection $U \to U_\lambda$ to each element of $\beta(t)$ and throw away infinitely many Q's to get $\beta_\lambda(t)$. Pick $\lambda_0 \geq 1$ such that $|z_0| < 1 - \frac{2}{\lambda_0}$. Set $\lambda_i = \lambda_0 + i$. Let m_i be the upper bound on the number of zeros inside $\{z : |z| < 1 - \frac{1}{\lambda_i}\}$. Thus we can define the path $\beta_i : [0,1] \to \frac{U_{\lambda_i}^{m_i}}{S_{m_i}}$. We want to inductively define a sequence of paths

$$\tilde{\beta}_i : [0, t_i] \to \overline{\Omega_A},$$

where intuitively each one draws nearer to P. First lift β_0 to a path $\tilde{\beta}_0 : [0,1] \to U_{\lambda_0}$. Note some coordinate of $\beta_0(0)$ is z_0 and all coordinates of $\beta_0(1)$ are Q, so we get the path $\tilde{\beta}_0$ from z_0 to Q in U_{λ_0}. Let t_0 be the smallest $t \in [0,1]$ such that $|\tilde{\beta}_0(t)| \geq 1 - \frac{2}{\lambda_0}$. Then by restricting the path to $[0, t_0]$ we get the path $\tilde{\beta}_0$ in $\mathbb{C}$ because

$$\left\{z \in U_{\lambda_0} : |z| \leq 1 - \frac{2}{\lambda_0}\right\}$$

is the same as

$$\left\{z \in \mathbb{C} : |z| \leq 1 - \frac{2}{\lambda_0}\right\}.$$

Finally, because of the projection we applied earlier,

$$\tilde{\beta}_0(t) \in \overline{\Omega_A},\ \forall t \in [0, t_0].$$

This takes care of the base case.

By the same process we find for each $\lambda_i = \lambda_0 + i$ a path $\tilde{\beta}_i : [t_{i-1}, 1] \to U_{\lambda_i}$ such that $\tilde{\beta}_i(t_{i-1}) = \tilde{\beta}_{i-1}(t_{i-1})$. Let t_i be the smallest $t \geq t_{i-1}$ such that

$$|\tilde{\beta}_i(t)| \geq 1 - \frac{2}{\lambda_i}$$

and obtain the path

$$\tilde{\beta}_i : [t_{i-1}, t_i] \to \overline{\Omega_A}.$$

Now append this to $\tilde{\beta}_{i-1}$ to get $\tilde{\beta}_i : [0, t_i] \to \overline{\Omega_A}$ such that $\tilde{\beta}_i(t)$ is always a coordinate of $\beta(t)$.

Now consider letting $t_\infty = \sup_i t_i$ and piece together all $\tilde{\beta}_i$'s. What we get is the continuous map

$$\tilde{\beta} : \big[0, t_\infty) \to \overline{\Omega_A}.$$

Let I be the set of limit points of $|\tilde{\beta}(t)|$ as $t \to t_\infty$. Clearly I is closed, as it contains all of its limit points. If I contains two different points a and b, then $|\beta(t)|$ approaches a and b infinitely often as $t \to t_\infty$, and thus it must contain all points between a and b. So I is a closed interval. Furthermore, as $\lim \tilde{\beta}(t_i) = 1$, we see that $1 \in I$.

Consider $\beta(t_\infty) = \{z_1, z_2, \ldots\}$. Let us assume there is a limit point, $r < 1$, and distinct from $|z_1|, |z_2|, \ldots$. Since $|z_i| \to 1$ and β is continuous, then r differs by some $\epsilon > 0$ from all $|z_i|$ in the neighborhood of t_∞. Thus r cannot be a limit point, and hence $r \notin I$, and hence $I \subset \{1, |z_1|, |z_2|, \ldots\}$. But since I is a closed interval, and I contains 1, we see that I must be the singleton 1. So all limit points of $\tilde{\beta}(t)$, $t \to t_\infty$ lie on the unit circle.

Case 1 Assume 1 is the only limit point of $\tilde{\beta}(t)$, $t \to t_\infty$. Then $\tilde{\beta}$ extends to a path $[0, t_\infty] \to \overline{\Omega_A}$ from z_0 to 1.

Case 2 Assume -1 is the only limit point of $\tilde{\beta}(t)$, $t \to t_\infty$. Then $\tilde{\beta}$ extends to a path $[0, t_\infty] \to \overline{\Omega_A}$ from z_0 to -1.

Case 3 Assume there is a limit point ,ψ, of $\tilde{\beta}(t)$ as $t \to t_\infty$ such that $\psi \neq \pm 1, |\psi| = 1$. We proved earlier that for all such points there exists an open neighborhood in $\overline{\Omega}$ centered at ψ. Since for some $t < t_\infty$, $\tilde{\beta}(t)$ is in this neighborhood, we can replace the tail end of $\tilde{\beta}$ on $[t, t_\infty)$ with a straight line from $\tilde{\beta}(t) \to \psi \in \overline{\Omega_A}$.

Case 4 Assume the only limit points of $\tilde{\beta}(t)$ are $1, -1$. But $\tilde{\beta}(t)$ is a map from $[0, t_\infty)$ to $\overline{\Omega_A}$ and thus must go between -1 and 1 infinitely often, as $t \to t_\infty$. It follows that there must be another limit point on the unit circle which is not ± 1. Thus this case reduces to Case 3.

In all of the cases we can connect z_0 to a point on the unit circle by a path $\tilde{\beta}$, completely contained in $\overline{\Omega_A}$. Since z_0 is arbitrary, this proves that $\overline{\Omega_A} \cap \{z : |z| \leq 1\}$ is path connected. Furthermore since the inverse map, $z \to \frac{1}{z}$ is closed and continuous, path connectivity must also hold for $\overline{\Omega_A} \cap \{z : |z| \geq 1\}$. Finally since these sets meet at the unit circle, which is path connected and contained in $\overline{\Omega_A}$ then it must follow that $\overline{\Omega_A}$ is path connected. Thus by the definition of A, $\overline{\Omega_n}$ and $\overline{\Omega_{n,0}}$ are path connected. □

5. An example of a non-connected set

In [17], Odlyzko and Poonen discuss how to draw the set $\Omega_{1,0}$. These methods and calculations are easily extended to Ω_A for any finite set A. The main observation to make is,

Lemma 5.1. *Let A be a finite subset of $\mathbb{C}$. Define $a = \max_{a_i \in A} |a_i|$. If $|z| < 1$ and $z \in \Omega_A$ then for all $n \geq 0$ there exists $a_i \in A$ such that*

$$\left| \sum_{i=0}^{n} a_i z^i \right| \leq a \frac{|z|^{n+1}}{1 - |z|}.$$

Proof. Notice if $z \in \Omega_A$ then there exists a_i such that

$$\sum_{i=0}^{\infty} a_i z^i = 0.$$

This implies that

$$\begin{aligned} \left| \sum_{i=0}^{n} a_i z^i \right| &= \left| - \sum_{i=n+1}^{\infty} a_i z^i \right| \\ &\leq \sum_{i=n+1}^{\infty} \left| a_i z^i \right| \\ &\leq a \sum_{i=n+1}^{\infty} |z|^i \\ &\leq a \frac{|z|^{n+1}}{1 - |z|}, \end{aligned}$$

which gives the desired result. □

This gives as an algorithm to determine which points are not in Ω_A. We simply pick some n reasonably large, and if

$$\min_{a_0, a_1, \cdots, a_n \in A} \left| \sum_{i=0}^{n} a_i z^i \right| > a \frac{|z|^{n+1}}{1 - |z|} \tag{2}$$

then $z \notin \Omega_A$. Computationally this allows us to find sets that are non-connected.

Theorem 5.1. *There exists an A such that Ω_A is not path connected. In particular $\Omega_{\{1,1+2i\}}$ is not path connected. (In fact $\Omega_{\{1,1+2i\}}$ is not even connected.)*

Proof. This is a proof by picture. Let $A = \{1, 1 + 2i\}$. Using Lemma 5.1 with $n = 4$ we get Figure 1. A better indication of what Ω_A looks

FIGURE 1

like in this case can be seen in Figures 2 and 3, (where Figure 3 is an enlargement of a section of Figure 2), which was done with the help of Fractint. (Code still needed to be written to test equation (2), Fractint was used only as a front end to allow easy movement about the diagram.) The different shadings in Figures 2 and 3 indicate which value of n was needed to eliminate the set. The curves indicate the boundary given by n to the equation

$$\min_{a_i \in A} \left| \sum_{i=0}^{n} a_i z^i \right| = a \frac{|z|^{n+1}}{1 - |z|}$$

In Figure 1, anything not contained within the curves cannot be in $\overline{\Omega_A}$, by the argument following the proof of Lemma 5.1. The crosses indicate some roots that are in Ω_A. In particular, they are roots of a degree 4 polynomials. These two things combine to show that Ω_A in this case cannot be connected. □

6. CONCLUSIONS, COMMENTS AND OPEN QUESTIONS

It is worth noting that Proposition 2.2 could have been made stronger. Given a power series $\sum a_i z^i$, $|a_0| = 1$ and $|a_i| \leq 1$, [11] gives a bound

Figure 2

Figure 3

on the number of roots of the power series within the disc of radius $1-\alpha$. In particular, it gives both upper and lower bounds of the form $c_1/\alpha \log(1/\alpha)$ and $\lfloor c_2/\alpha \log(1/\alpha) \rfloor$ respectively, for absolute constants, $c_1, c_2 > 0$. This bound is far sharper than Proposition 2.2. For our purposes it doesn't matter, as we just need some bound, we don't really care what it is.

There are a few questions that are raised as a result of this paper.

(1) For what other sets A is $\overline{\Omega_A}$ path connected?
(2) Do there exist A where $\overline{\Omega_A}$ is connected, but not path connected?
(3) What can be said if we instead looked at the closure of multiple roots (to some fixed multiplicity) of polynomials or power series with coefficients restricted in some way?
(4) What can be said about the Hausdorff dimension of the boundary of these sets?

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References

[1] Franck Beaucoup, Peter Borwein, David W. Boyd, and Christopher Pinner, *Power series with restricted coefficients and a root on a given ray*, Math. Comp. **67** (1998), no. 222, 715–736. MR **98k:**30006

[2] Frank Beaucoup, Peter Borwein, David W. Boyd, and Christopher Pinner, *Multiple roots of* $[-1,1]$ *power series*, J. London Math. Soc. (2) **57** (1998), no. 1, 135–147. MR **99c:**30005

[3] E. Bogomolny, O. Bohigas, and P. Leboeuf, *Quantum chaotic dynamics and random polynomials*, J. Statist. Phys. **85** (1996), no. 5-6, 639–679. MR **98a:**81046

[4] E. Bombieri and J. Vaaler, *On Siegel's lemma*, Invent. Math. **73** (1983), no. 1, 11–32. MR **85g:**11049a

[5] ———, *Addendum to: "On Siegel's lemma"*, Invent. Math. **75** (1984), no. 2, 377. MR **85g:**11049b

[6] Peter Borwein and Tamás Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois J. Math. **41** (1997), no. 4, 667–675. MR **98g:**30008

[7] Peter Borwein and Michael J. Mossinghoff, *Newman polynomials with prescribed vanishing and integer sets with distinct subset sums*, Math. Comp. **72** (2003), no. 242, 787–800 (electronic). MR **2003k:**11036
[8] Peter Borwein and Christopher Pinner, *Polynomials with* $\{0, +1, -1\}$ *coefficients and a root close to a given point*, Canad. J. Math. **49** (1997), no. 5, 887–915. MR **98m:**11014
[9] David W. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients. II*, Math. Comp. **71** (2002), no. 239, 1205–1217 (electronic). MR **2003d:**11035
[10] John Brillhart, Michael Filaseta, and Andrew Odlyzko, *On an irreducibility theorem of A. Cohn*, Canad. J. Math. **33** (1981), no. 5, 1055–1059. MR **83c:**12003
[11] Tamás Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. **49** (2001), no. 1, 97–111. MR **2002c:**30009
[12] P. Erdös and P. Turán, *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119. MR 11,431b
[13] Leopold Flatto, Jeffrey C. Lagarias, and Bjorn Poonen, *The zeta function of the beta transformation*, Ergodic Theory Dynam. Systems **14** (1994), no. 2, 237–266. MR **95c:**58141
[14] John Milnor, *Periodic orbits, externals rays and the Mandelbrot set: an expository account*, Astérisque (2000), no. 261, xiii, 277–333, Géométrie complexe et systèmes dynamiques (Orsay, 1995). MR **2002e:**37067
[15] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994. MR **95m:**30009
[16] Michael J. Mossinghoff, *Polynomials with restricted coefficients and prescribed non-cyclotomic factors*, LMS J. Comput. Math. (to appear).
[17] A. M. Odlyzko and B. Poonen, *Zeros of polynomials with* $0, 1$ *coefficients*, Enseign. Math. (2) **39** (1993), no. 3-4, 317–348. MR **95b:**11026
[18] Boris Solomyak, *Conjugates of beta-numbers and the zero-free domain for a class of analytic functions*, Proc. London Math. Soc. (3) **68** (1994), no. 3, 477–498. MR **95c:**30010
[19] E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, 1939.

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