

# On $\mathbb{Z}$ -Modules of Algebraic Integers

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*Abstract.* Let  $q$  be an algebraic integer of degree  $d \geq 2$ . Consider the rank of the multiplicative subgroup of  $\mathbb{C}^*$  generated by the conjugates of  $q$ . We say  $q$  is of *full rank* if either the rank is  $d - 1$  and  $q$  has norm  $\pm 1$ , or the rank is  $d$ . In this paper we study some properties of  $\mathbb{Z}[q]$  where  $q$  is an algebraic integer of full rank. The special cases of when  $q$  is a Pisot number and when  $q$  is a Pisot-cyclotomic number are also studied. There are four main results.

- (1) If  $q$  is an algebraic integer of full rank and  $n$  is a fixed positive integer, then there are only finitely many  $m$  such that  $\text{disc}(\mathbb{Z}[q^m]) = \text{disc}(\mathbb{Z}[q^n])$ .
- (2) If  $q$  and  $r$  are algebraic integers of degree  $d$  of full rank and  $\mathbb{Z}[q^n] = \mathbb{Z}[r^n]$  for infinitely many  $n$ , then either  $q = \omega r'$  or  $q = \text{Norm}(r)^{2/d} \omega / r'$ , where  $r'$  is some conjugate of  $r$  and  $\omega$  is some root of unity.
- (3) Let  $r$  be an algebraic integer of degree at most 3. Then there are at most 40 Pisot numbers  $q$  such that  $\mathbb{Z}[q] = \mathbb{Z}[r]$ .
- (4) There are only finitely many Pisot-cyclotomic numbers of any fixed order.

## 1 Introduction

In this paper we study some properties of  $\mathbb{Z}[q]$ , where  $q$  is an algebraic integer of full rank. As well, we study the special cases of Pisot numbers and Pisot-cyclotomic numbers. We begin by recalling the definition of full rank.

**Definition** Let  $q$  be an algebraic integer of degree  $d \geq 2$ . Consider the rank of the multiplicative subgroup of  $\mathbb{C}^*$  generated by the conjugates of  $q$ . We say  $q$  is of *full rank* if either the rank is  $d - 1$  and  $q$  has norm  $\pm 1$ , or the rank is  $d$ .

It is worth observing that if the multiplicative subgroup of  $\mathbb{C}^*$  generated by the conjugates of  $q$  has rank  $d$ , then  $q$  cannot have norm  $\pm 1$ . As we later show, an important class of full rank algebraic integers is given by the collection of Pisot numbers.

**Definition** A *Pisot number* is a real algebraic integer greater than 1, all of whose conjugates are of modulus strictly less than 1.

Our first result is the following theorem.

**Theorem 1.1** For a given algebraic integer  $q$  of full rank and a fixed positive integer  $n$ , there are only finitely many  $m$  for which  $\text{disc}(\mathbb{Z}[q^m]) = \text{disc}(\mathbb{Z}[q^n])$ . Hence there are only finitely many  $m$  such that  $\mathbb{Z}[q^m] = \mathbb{Z}[q^n]$ .

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From this we obtain the following corollary.

**Corollary 1.2** *Let  $q$  be an algebraic integer of full rank. Then the absolute value of the discriminant of  $q^n$  tends to infinity as  $n \rightarrow \infty$ .*

We next consider under which conditions on two algebraic integers  $q$  and  $r$  we can have  $\mathbb{Z}[q^n] = \mathbb{Z}[r^n]$  for infinitely many  $n$ .

**Theorem 1.3** *Let  $q$  and  $r$  be full rank algebraic integers of degree  $d$ . If  $\mathbb{Z}[q^n] = \mathbb{Z}[r^n]$  for infinitely many  $n$ , then either  $q = \omega r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ , or  $q = \text{Norm}(r)^{2/d} \omega / r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ .*

Relating this theorem to Pisot numbers gives the following result.

**Corollary 1.4** *Let  $q$  and  $r$  be Pisot numbers. Suppose that  $\mathbb{Z}[q^n] = \mathbb{Z}[r^n]$  for infinitely many  $n$ . Then  $q = r$ .*

For Pisot numbers  $q$  and  $r$  of small degree we can in fact obtain upper bounds on the number of  $n$  for which  $\mathbb{Z}[q^n] = \mathbb{Z}[r^n]$ .

**Theorem 1.5** *Let  $r$  be an algebraic integer of degree at most 3. Then there are at most 40 Pisot numbers  $q$  such that  $\mathbb{Z}[q] = \mathbb{Z}[r]$ .*

It should be mentioned that in practice no example has been found of an algebraic integer  $r$  of degree 3 for which there are more than 7 Pisot numbers  $q$  satisfying  $\mathbb{Z}[q] = \mathbb{Z}[r]$ .

An important application of Theorem 1.5 is the determination of Pisot-cyclotomic numbers.

**Definition** A *Pisot-cyclotomic number* of order  $n$  is a Pisot number  $q$  such that  $\mathbb{Z}[q] = \mathbb{Z}[\beta_n]$ , where  $\beta_n = 2 \cos\left(\frac{2\pi}{n}\right)$ .

Pisot-cyclotomic numbers have applications to the study of quasicrystals and quasilattices [3, 4]. Methods to find Pisot-cyclotomic numbers of higher orders require solutions to homogeneous Diophantine problems in several variables and are the obvious extension to Theorem 1.5. This is discussed further in [1].

The last result of this paper implies that there are only finitely many Pisot-cyclotomic numbers of order  $n$  for any fixed  $n$ . In fact, a stronger result is given.

**Theorem 1.6** *Let  $r$  be a Pisot number with the property that all of its conjugates lie in the extension  $\mathbb{Q}(r)$ . Then there are only finitely many Pisot numbers  $q$  with the property that  $\mathbb{Z}[q] = \mathbb{Z}[r]$ .*

**Corollary 1.7** *There are only finitely many Pisot-cyclotomic numbers of any fixed order.*

Theorem 1.1 is proved in Section 4. Theorem 1.3, along with some interesting results concerning unitary matrices, is shown in Section 5. Section 6 gives a proof of Theorem 1.5. Section 7 gives of proof of Theorem 1.6 using the Schmidt Subspace Theorem. Section 8 lists some possible future work and some open problems in this area.

## 2 Background on the Discriminant

Throughout this paper we use the discriminant of an algebraic integer in our considerations. Given an algebraic integer  $q$  with conjugates  $q = q_0, q_1, \dots, q_{d-1}$ , we define the *discriminant* of  $q$  to be

$$\text{disc}(q) := \prod_{0 \leq i < j < d} (q_i - q_j)^2 = \left( \det(q_{i-1}^{j-1})_{i,j=1}^d \right)^2.$$

The following theorem is a famous result which shows the utility of discriminants.

**Theorem 2.1** *Let  $q$  and  $r$  be algebraic integers. If  $\mathbb{Z}[q] = \mathbb{Z}[r]$ , then  $\text{disc}(q) = \text{disc}(r)$ . Conversely, if  $q$  and  $r$  are algebraic integers of the same degree with  $\text{disc}(q) = \text{disc}(r)$  and  $\mathbb{Z}[q] \subset \mathbb{Z}[r]$ , then  $\mathbb{Z}[q] = \mathbb{Z}[r]$ .*

**Proof** cf. Marcus [6, Theorem 11 and exercise 27 on page 45]. ■

From the first part of the Theorem, we observe that we can define  $\text{disc}(\mathbb{Z}[q]) = \text{disc}(q)$  where  $q$  is any algebraic integer that generates the ring.

## 3 Algebraic Integers of Full Rank

In this section we prove some important facts about algebraic integers of full rank.

**Proposition 3.1** *Every Pisot number is of full rank.*

**Proof** Let  $q$  be a Pisot number and let  $q = q_0, \dots, q_{d-1}$  denote its conjugates. Let  $m$  denote the norm of  $q$ . Suppose that

$$(3.1) \quad q_0^{a_0} \cdots q_{d-1}^{a_{d-1}} = 1.$$

Let  $j = \max\{-a_i \mid 0 \leq i < d\}$ . Then  $j + a_i \geq 0$  for  $0 \leq i < d$  and there exists some  $k$  such that  $j + a_k = 0$ . Then multiplying both sides of equation (3.1) by  $(q_0 \cdots q_{d-1})^j$  we get

$$(3.2) \quad q_0^{j+a_0} \cdots q_{d-1}^{j+a_{d-1}} = m^j.$$

Now since the Galois group of  $\mathbb{Q}(q_0, \dots, q_{d-1})/\mathbb{Q}$  acts transitively on the conjugates of  $q$ , there is an automorphism  $\sigma$  which sends  $q_k$  to  $q = q_0$ . Applying  $\sigma$  to both sides of equation (3.2), we see that there exist nonnegative integers  $b_1, \dots, b_{d-1}$  such that  $q_1^{b_1} \cdots q_{d-1}^{b_{d-1}} = m^j$ . Since  $q$  is a Pisot number,  $|q_1|, \dots, |q_{d-1}| < 1$ . Since  $|m| \geq 1$ , we see that  $b_1 = b_2 = \cdots = b_{d-1} = 0$  and  $|m| = 1$ . Thus if the norm of  $q$  is not  $\pm 1$ , then the multiplicative group generated by the conjugates of  $q$  has rank  $d$ . If the norm is equal to  $\pm 1$ , then the only relations in the multiplicative group are of the form  $(q_0 \cdots q_{d-1})^j = 1$ . Hence the multiplicative group has rank  $d - 1$ . Thus  $q$  has full rank. ■

**Proposition 3.2** *Let  $q$  be an algebraic integer of full rank having conjugates  $q = q_0, q_1, \dots, q_{d-1}$ . Suppose for some integers  $a_0, \dots, a_{d-1}$  satisfying either*

- *at least one of the  $a_i$  is 0; or*
- *the  $a_i$  sum to 0,*

*that  $\prod_i q_i^{a_i}$  is a root of unity. Then all the  $a_i$  are 0.*

**Proof** By replacing each  $a_i$  by  $ma_i$  for some appropriate  $m$ , we may assume that the root of unity is in fact 1. If the norm of  $q$  is not  $\pm 1$ , then since the multiplicative group generated by  $q = q_0, \dots, q_{d-1}$  has rank  $d$ , we conclude that  $a_0 = \dots = a_{d-1} = 0$ .

Thus we may assume that  $q$  has norm  $\pm 1$ . Let  $G$  denote the multiplicative group generated by  $q_0, \dots, q_{d-1}$ . Then we have a surjection  $\mathbb{Z}^d \rightarrow G$ . By assumption, the kernel of this map is a subgroup of rank 1. Since it is finitely generated and torsion free the kernel is isomorphic to  $\mathbb{Z}$ . Notice that  $q_0^2 \cdots q_{d-1}^2 = 1$  and so by the above remarks if  $q_0^{a_0} \cdots q_{d-1}^{a_{d-1}} = 1$ , then  $a_0 = a_1 = \dots = a_{d-1}$ . Thus if  $a_i = 0$  for some  $i$  or  $a_0 + \dots + a_{d-1} = 0$ , we must have that all the  $a_i$  are 0. ■

It should be noted that Proposition 3.2 is true if we replace algebraic integers with algebraic numbers. This generality was not needed for this paper.

#### 4 Proof of Theorem 1.1.

Throughout the rest of the paper, we take  $S_m$  to be the set of permutations of  $\{0, 1, \dots, m - 1\}$ .

An important result we use is the so-called Skolem–Mahler–Lech theorem. See, for example, Lech [5].

**Theorem 4.1 (Skolem–Mahler–Lech)** *Suppose that a rational function over a field of characteristic 0 whose series expansion  $\sum_{i=0}^{\infty} c_i z^i$  has infinitely many  $c_i = 0$ . Then there exist integers  $a, b$  with  $0 \leq b < a$  such that  $c_{am+b} = 0$  for all  $m$  sufficiently large.*

**Corollary 4.2** *Suppose that a rational function over a field of characteristic 0 whose series expansion  $f(z) := \sum_{i=0}^{\infty} c_i z^i$  has infinitely many  $c_i = C$  for some constant  $C$ . Then there exist integers  $a, b$  with  $0 \leq b < a$  such that  $c_{am+b} = C$  for all  $m$  sufficiently large.*

**Proof** Consider  $g(z) = f(z) - \frac{C}{1-z}$ . ■

**Lemma 4.3** *Let  $q$  be an algebraic integer with conjugates  $q = q_0, \dots, q_{d-1}$  and let  $A_n$  denote the  $d \times d$  matrix whose  $(i, j)$  entry is  $q_{i-1}^{(j-1)n}$ . Then for  $0 \leq b < a$ , the power series  $\sum_{n=1}^{\infty} \det(A_{an+b})z^n$  is a rational function whose poles are of the form  $\prod_{i=0}^{d-1} q_i^{-a\sigma(i)}$  with  $\sigma \in S_d$ .*

**Proof** Using the Vandermonde formula for  $\det(A_n)$ , we see

$$\begin{aligned}
F(z) &:= \sum_{n=0}^{\infty} \det(A_{an+b}) z^n \\
&= \sum_{\sigma \in S_d} \sum_{n=0}^{\infty} \operatorname{sgn}(\sigma) (q_0^{\sigma(0)} \cdots q_{d-1}^{\sigma(d-1)})^{an+b} z^n \\
&= \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) (q_0^{\sigma(0)} \cdots q_{d-1}^{\sigma(d-1)})^b \sum_{n=0}^{\infty} (q_0^{\sigma(0)} \cdots q_{d-1}^{\sigma(d-1)})^{an} z^n \\
&= \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) (q_0^{\sigma(0)} \cdots q_{d-1}^{\sigma(d-1)})^b \left( \frac{1}{1 - q_0^{a\sigma(0)} \cdots q_{d-1}^{a\sigma(d-1)} z} \right).
\end{aligned}$$

Hence  $F(z)$  is a rational function whose poles are of the form  $\prod_i q_i^{-a\sigma(i)}$  with  $\sigma \in S_d$ .  $\blacksquare$

Again, Lemma 4.3 is valid with algebraic integers replaced with algebraic numbers, but this generality was not needed for this paper.

**Proof of Theorem 1.1** Let  $q = q_0, q_1, \dots, q_{d-1}$  denote the conjugates of  $q$  and let  $A_n$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $q_{i-1}^{n(j-1)}$ . We note that  $q_i^n \neq q_j^n$  for distinct conjugates  $q_i$  and  $q_j$  of  $q$  since  $q$  has full rank. Hence

$$\det(A_n)^2 = \operatorname{disc}(\mathbb{Z}[q^n])$$

Suppose there is an infinite subset  $\mathcal{S}$  of  $\mathbb{N}$  such that  $\operatorname{disc}(\mathbb{Z}[q^n])$  is constant for  $n \in \mathcal{S}$ . By Theorem 2.1, there is some constant  $C$  such that  $\det(A_n) = C$  for infinitely many  $n \in \mathcal{S}$ . Let

$$F(z) := \sum_{n=0}^{\infty} \det(A_n) z^n.$$

Then by Lemma 4.3,  $F(z)$  is a rational function. Now suppose that  $\det(A_n)$  is equal to  $C$  for infinitely many  $n$ . By Theorem 4.1, there exist  $a, b$  such that  $\det(A_{am+b}) = C$  for all  $m$  sufficiently large. Hence

$$G(z) := \sum_{m=0}^{\infty} \det(A_{am+b}) z^m = P(z) + C/(1-z),$$

for some polynomial  $P(z)$ . By Lemma 4.3,  $G(z)$  is a rational function whose poles are of the form  $\prod_i q_i^{-a\sigma(i)}$  with  $\sigma \in S_d$ . Since  $\sigma(i) = 0$  for some  $i$  and  $q$  has full rank, we see that none of the poles are roots of unity by Proposition 3.2. Consequently, for a given integer  $k$  there are at most finitely many  $n$  for which  $\operatorname{disc}(\mathbb{Z}[q^n]) = k$ .  $\blacksquare$

### 5 Proof of Theorem 1.3

Throughout this section  $q$  and  $r$  are algebraic integers of full rank, and degree  $d$ . We shall also use  $q = q_0, \dots, q_{d-1}$  and  $r = r_0, \dots, r_{d-1}$  to denote the conjugates of  $q$  and  $r$  respectively. For convenience, we take  $S_d$  to be the group of permutations of  $\{0, 1, \dots, d-1\}$ . We define

$$X(\sigma) = \prod_{j=0}^{d-1} q_j^{\sigma(j)} \quad \text{and} \quad Y(\sigma) = \prod_{j=0}^{d-1} r_j^{\sigma(j)}$$

and we define

$$v(\sigma) := (\sigma(0), \dots, \sigma(d-1)) \in \mathbb{Q}^{1 \times d}.$$

Finally, we let  $\mathcal{P}_d = \{v(\sigma) \mid \sigma \in S_d\}$ .

**Lemma 5.1** *The  $\mathbb{Q}$ -vector space spanned by  $\mathcal{P}_d$  is  $\mathbb{Q}^{1 \times d}$ ; moreover, a basis is given by  $\{v(\text{id})\} \cup \{v((i, i+1)) \mid 0 \leq i \leq d-2\}$ .*

**Proof** We claim that

$$\mathcal{S} = \{v((i, i+1)) - v(\text{id}) \mid 0 \leq i \leq d-2\}$$

is linearly independent. To see this, suppose that

$$\sum_{i=0}^{d-2} c_i (v((i, i+1)) - v(\text{id})) = (0, 0, \dots, 0).$$

Then for  $0 \leq j \leq d-1$  we have

$$\sum_{i=0}^{d-2} c_i (\delta_{j,i} - \delta_{j,i+1}) = 0.$$

Taking  $j = 0$ , we see that  $c_0 = 0$ . We also have  $c_j - c_{j-1} = 0$  for  $1 \leq j \leq d-2$  and so  $c_0 = \dots = c_{d-2} = 0$ . Thus  $\mathcal{S}$  is linearly independent. Notice that the sum of the entries of each element of  $\mathcal{S}$  is equal to 0 and hence  $v(\text{id})$  cannot possibly be in the span of  $\mathcal{S}$ . It follows that

$$\{v((i, i+1)) \mid 0 \leq i \leq d-2\} \cup \{v(\text{id})\}$$

is a basis for  $\mathbb{Q}^{1 \times d}$ . ■

**Lemma 5.2** *Suppose that for every  $\sigma \in S_d$  there is some  $\tau \in S_d$  such that  $X(\sigma) = Y(\tau)$ . Then there is a  $d \times d$  matrix  $E$  with rational entries such that whenever  $X(\sigma) = Y(\tau)$ , we have  $v(\tau) = v(\sigma)E$ ; moreover the row sums of  $E$  are all equal to 1.*

**Proof** Take a basis  $\{v(\sigma_0), \dots, v(\sigma_{d-1})\}$  for the  $\mathbb{Q}$ -vector space spanned by  $\mathcal{P}_d$ . Then we can find  $\tau_0, \dots, \tau_{d-1}$  such that  $X(\sigma_i) = Y(\tau_i)$  for  $0 \leq i < d$ . Since  $\{v(\sigma_i) \mid 0 \leq i < d\}$  is a basis, there is a unique matrix  $E$  such that  $v(\tau_i) = v(\sigma_i)E$  for  $0 \leq i < d$ . Let  $\sigma \in S_d$ . Then there exist integers  $a_0, \dots, a_{d-1}$  and  $b \neq 0$  such that

$$(5.1) \quad bv(\sigma) = a_0v(\sigma_0) + \dots + a_{d-1}v(\sigma_{d-1}).$$

Right-multiplying both sides of equation (5.1) by the vector  $(1, 1, \dots, 1)^T$ , we see that

$$b \binom{d}{2} = (a_0 + \dots + a_{d-1}) \binom{d}{2}.$$

Right-multiplying both sides of equation (5.1) by  $E$ , we see

$$bv(\sigma)E = a_0v(\tau_0) + \dots + a_{d-1}v(\tau_{d-1}).$$

Write  $bv(\sigma)E = (c_0, \dots, c_{d-1})$ . Then

$$c_0 + \dots + c_{d-1} = (a_0 + \dots + a_{d-1}) \binom{d}{2} = b \binom{d}{2}.$$

Now

$$\prod_{i=0}^{d-1} r_i^{c_i} = \prod_{i=0}^{d-1} Y(\tau_i)^{a_i} = \prod_{i=0}^{d-1} X(\sigma_i)^{a_i} = X(\sigma)^b.$$

By assumption there is some  $\tau$  such that  $X(\sigma) = Y(\tau)$ , and so

$$\prod_{i=0}^{d-1} r_i^{c_i} = \prod_{i=0}^{d-1} r_i^{b\tau(i)}.$$

Equivalently,

$$\prod_{i=0}^{d-1} r_i^{c_i - b\tau(i)} = 1.$$

Notice that

$$\sum_{i=0}^{d-1} (c_i - b\tau(i)) = \sum_{i=0}^{d-1} c_i - b \binom{d}{2} = 0.$$

Thus,  $c_i = b\tau(i)$  for all  $i$  by Proposition 3.2, and so we see that  $v(\tau) = v(\sigma)E$  for each  $\sigma \in S_d$ .

To show that the row sums of  $E$  are all equal to 1, we use our basis of row vectors. Using the fact that  $v(\tau) = v(\sigma)E$ , we see

$$\begin{bmatrix} v(\sigma_0) \\ \vdots \\ v(\sigma_{d-1}) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} cv(\tau_0) \\ \vdots \\ v(\tau_{d-1}) \end{bmatrix} E \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

Furthermore, we have

$$\begin{bmatrix} v(\sigma_0) \\ \vdots \\ v(\sigma_{d-1}) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} v(\tau_0) \\ \vdots \\ v(\tau_{d-1}) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \binom{d}{2} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

From this we get

$$\begin{bmatrix} v(\tau_0) \\ \vdots \\ v(\tau_{d-1}) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} v(\tau_0) \\ \vdots \\ v(\tau_{d-1}) \end{bmatrix} E \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

Since  $q$  and  $r$  are of full rank, we see that  $\{v(\tau_i) \mid 0 \leq i \leq d-1\}$  is a basis for  $\mathbb{Q}^{d \times 1}$ . Hence the matrix whose  $i$ -th row is  $v(\tau_i)$  is invertible and so we see

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = E \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

from which the result follows. ■

We are now almost ready to prove our key structure result for matrices which send  $\mathcal{P}_n$  to itself. We need a few simple lemmas before we can continue.

**Lemma 5.3** *Let  $d$  be a positive integer and let  $\sigma \in S_d$ . Then  $\langle v(\text{id}), v(\sigma) \rangle = \langle v(\text{id}), v(\text{id}) \rangle - 1$  if and only if  $\sigma = (i, i+1)$  for some  $i$  with  $0 \leq i \leq d-2$ .*

**Proof** When  $\sigma = (i, i+1)$ , it is easy to verify that  $\langle v(\text{id}), v(\sigma) \rangle = \langle v(\text{id}), v(\text{id}) \rangle - 1$ . We prove the other direction by induction. The claim is clearly true when  $d = 1$ . Suppose that the claim is true when  $d = m$  and consider the case  $d = m+1$ . If  $\sigma(m) = m$ , then by the inductive hypothesis we have  $\sigma = (i, i+1)$  for some  $i < m-1$ . Thus we may assume that  $\sigma(m) = j < m$ . Also, there is some  $k < m$  with  $\sigma(k) = m$ . Let  $S = 1^2 + \cdots + m^2$ . Then

$$\begin{aligned} S - 1 &= \langle v(\text{id}), v(\sigma) \rangle \\ &= \langle (0, 1, \dots, m), (\sigma(0), \dots, \sigma(m)) \rangle \\ &= km + mj + \sum_{i \neq k, m} i\sigma(i) \\ &\leq km + mj + \sum_{i \neq k, m} \frac{i^2 + \sigma(i)^2}{2} \\ &= km + mj - k^2/2 - j^2/2 - m^2 + S \\ &= -(m-k)^2/2 - (m-j)^2/2 + S. \end{aligned}$$

Hence  $(m-k)^2 + (m-j)^2 \leq 2$ . Since  $j, k < m$  are integers we must have equality and  $k = j = m-1$ . Thus  $\sigma(m) = m-1$  and  $\sigma(m-1) = m$ . Thus  $\sigma$  restricted to

$\{0, 1, \dots, m-2\}$  is a permutation of this set. Using the Cauchy–Schwarz inequality we see

$$\begin{aligned} \langle v(\text{id}), v(\sigma) \rangle &= \langle (0, 1, \dots, m-2), (\sigma(0), \dots, \sigma(m-2)) \rangle + 2(m-1)m \\ &\leq (0^2 + 1^2 + \dots + (m-2)^2) + 2(m-1)m = S-1. \end{aligned}$$

By assumption,  $\langle v(\text{id}), v(\sigma) \rangle = S-1$  and so we must have an equality in the Cauchy–Schwarz inequality. Hence  $\sigma(i) = i$  for  $i < m-1$  and  $\sigma$  is just the transposition  $(m-1, m)$ . This proves the lemma. ■

**Lemma 5.4** *Let  $E$  be an orthogonal  $d \times d$  matrix with the property that  $\mathcal{P}_d E = \mathcal{P}_d$  and  $v(\text{id})E = v(\text{id})$ . Then either  $E$  is the identity matrix, or  $E = \frac{2}{d}J - U$ , where  $U$  is the permutation matrix whose  $(i, j)$  entry is  $\delta_{i+j, d+1}$  and  $J$  is the matrix of all 1's.*

**Proof** First observe that if  $E = \frac{2}{d}J - U$  then

$$EE^T = \left( \frac{2}{d}J - U \right) \left( \frac{2}{d}J - U^{-1} \right) = \frac{4}{d^2}J^2 - \frac{2}{d}JU^{-1} - \frac{2}{d}UJ + I = I,$$

and so  $E$  is unitary. Furthermore,

$$\begin{aligned} (5.2) \quad v(\sigma)E &= \frac{2}{d}v(\sigma)J - v(\sigma)U \\ &= (d-1, d-1, \dots, d-1) - (\sigma(d-1), \dots, \sigma(0)) \\ &= (d-1 - \sigma(d-1), d-1 - \sigma(d-2), \dots, d-1 - \sigma(0)) \\ &= v(\tau) \end{aligned}$$

for some permutation  $\tau$ . Thus  $E = \frac{2}{d}J - U$  satisfies the conditions of the lemma.

We now look at which matrices satisfy the conditions of the statement of the lemma.

$$\begin{aligned} \langle v(\text{id}), v(\text{id}) \rangle - 1 &= \langle v(\text{id}), v((i, i+1)) \rangle = \langle v(\text{id})E, v((i, i+1))E \rangle \\ &= \langle v(\text{id}), v((i, i+1))E \rangle. \end{aligned}$$

Thus,  $v((i, i+1))E = v((j, j+1))$  for some  $j$  by Lemma 5.3. Let  $\tau$  be a permutation of  $\{0, 1, \dots, d-2\}$  such that  $v((i, i+1))E = v((\tau(i), \tau(i)+1))$ .

Let  $e_i$  denote the row vector which has a 1 in the  $i$ -th position and zeros everywhere else. Then  $v((i, i+1)) - v(\text{id}) = e_{i+1} - e_{i+2}$ . Consequently,

$$\langle v((i, i+1)) - v(\text{id}), v((j, j+1)) - v(\text{id}) \rangle = 0 \quad \text{if and only if} \quad |i - j| \geq 2.$$

Since  $E$  is unitary and  $v(\text{id})E = v(\text{id})$ , we see that

$$\begin{aligned} &\langle v((i, i+1)) - v(\text{id}), v((j, j+1)) - v(\text{id}) \rangle \\ &= \langle v((i, i+1))E - v(\text{id})E, v((j, j+1))E - v(\text{id})E \rangle \\ &= \langle v(\tau(i), \tau(i)+1) - v(\text{id}), v(\tau(j), \tau(j)+1) - v(\text{id}) \rangle. \end{aligned}$$

It is then clear that  $|\tau(i) - \tau(j)| \geq 2$  if and only if  $|i - j| \geq 2$ . Hence  $|\tau(i) - \tau(j)| \leq 1$  if and only if  $|i - j| \leq 1$ . Moreover, since  $\tau$  is a permutation, if  $i \neq j$ , then  $\tau(i) \neq \tau(j)$ , and so  $|i - j| = 1$  if and only if  $|\tau(i) - \tau(j)| = 1$  for  $0 \leq i, j \leq d - 2$ .

Notice that 0 and  $d - 2$  are the only values of  $i \leq d - 2$  such that there is exactly one  $j$  between 0 and  $d - 2$  with  $|i - j| = 1$ . Thus  $\tau(0) \in \{0, d - 2\}$ . There are now two cases. Suppose that  $\tau(0) = 0$  and that  $\tau(i) \neq i$  for some  $i$ . Pick  $i_0 > 0$  minimal with  $\tau(i_0) \neq i_0$ . Then

$$1 = |\tau(i_0) - \tau(i_0 - 1)| = |\tau(i_0) - (i_0 - 1)|.$$

Hence  $\tau(i_0) \in \{i_0 - 2, i_0\}$ . If  $i_0 - 2 \geq 1$ , then  $\tau(i_0 - 2) = i_0 - 2$  and so  $\tau(i_0) = i_0$ ; if  $i_0 - 2 < 1$ , then  $\tau(i_0)$  cannot equal  $i_0 - 2$  and hence must be  $i_0$ , a contradiction. It follows that if  $\tau(0) = 0$  then  $\tau$  is the identity. A similar argument shows that if  $\tau(0) = d - 2$ , then  $\tau(i) = d - 2 - i$  for  $0 \leq i \leq d - 2$ .

If  $\tau$  is the identity, then  $v((i, i + 1))E = v((i, i + 1))$  for all applicable  $i$  and  $v(\text{id})E = v(\text{id})$  and hence by Lemma 5.1,  $E$  is the identity matrix, and so the claim is true in this case.

If  $\tau$  is not the identity, then  $v((i, i + 1))E = v((d - i - 2, d - i - 1))$  for all applicable  $i$  and  $v(\text{id})E = v(\text{id})$ . Above, we saw that  $X = \frac{2}{d}J - U$  satisfies:

- $\mathcal{P}_d X = \mathcal{P}_d$ ;
- $X$  is orthogonal; and
- $v((i, i + 1))X = v(d - i - 2, d - i - 1)$  for all applicable  $i$  and  $v(\text{id})X = v(\text{id})$ .

Thus we see that in this case  $E = \frac{2}{d}J - U$ . The result follows. ■

**Proposition 5.5** *Let  $G = \{Y \in \text{GL}_d(\mathbb{C}) \mid \mathcal{P}_d Y = \mathcal{P}_d\}$ . Then  $G \cong \mathbb{Z}/2\mathbb{Z} \times S_d$ , where  $S_d$  corresponds to the group of permutation matrices and  $\mathbb{Z}/2\mathbb{Z}$  corresponds to the group  $\{\mathbf{I}_d, \frac{2}{d}J - \mathbf{I}_d\}$ , where  $J$  is the matrix of all 1's.*

**Proof** Clearly  $G$  is a finite group and the set of permutation matrices is a subgroup of  $G$ . Since  $G$  is a finite linear group, there is an invertible matrix  $M$  such that  $MYM^{-1}$  is unitary for all  $Y \in G$ . In particular  $MPM^{-1}$  is unitary for all permutation matrices  $P$ . Hence if  $P$  is a permutation matrix,

$$\mathbf{I}_d = (MPM^{-1})(MPM^{-1})^* = (MPM^{-1})((M^{-1})^* P^* M^*).$$

Letting  $X = M^{-1}(M^{-1})^*$ , we see that  $PXP^* = X$ ; moreover, since  $P$  is unitary, we have  $XP = PX$  for all permutation matrices  $P$ . The permutation matrices arise from a representation of  $S_d$  which is a direct sum of two irreducible representations. It follows that the set of  $X$  which commute with all permutation matrices is a 2 dimensional  $\mathbb{C}$ -vector space. Since both  $\mathbf{I}_d$  and  $J$  commute with every permutation matrix, we see that these two matrices span the vector space of matrices which commute with every permutation matrix. Hence  $X = \alpha\mathbf{I}_d + \beta J$ . Since  $X$  is invertible and  $J$  is not, we see that  $\alpha$  is nonzero. Let  $Y \in G$ . Observe that for each  $\sigma \in S_d$ , there exists  $\tau \in S_d$  such that  $v(\sigma)Y = v(\tau)$ . Let  $y_i$  denote the sum of the  $i^{\text{th}}$  row of  $Y$ . Then multiplication by  $(1, 1, \dots, 1)^T$  gives

$$v(\sigma) \cdot (y_1, \dots, y_d) = \begin{pmatrix} d \\ 2 \end{pmatrix}.$$

In particular,  $v(\sigma) \cdot (y_1 - 1, \dots, y_d - 1) = 0$  for all  $\sigma \in S_d$ . Since the vectors  $v(\sigma)$  span  $\mathbb{Q}^d$ , we see that the row sums of  $Y$  must all be equal to 1. We have

$$\mathbf{I}_d = (MYM^{-1})(M^{-1})^*Y^*M^*,$$

and hence  $YXY^* = X$ . Equivalently,  $Y(\alpha\mathbf{I}_d + \beta J)Y^* = \alpha\mathbf{I}_d + \beta J$ . Since  $Y$  has row sums equal to 1 and  $Y^*$  has column sums equal to 1, we see that  $YJY^* = J$ , and so we conclude that  $\alpha YY^* = \alpha\mathbf{I}_d$ . Since  $\alpha \neq 0$ , we see that  $Y$  is unitary. Thus  $G$  is a unitary group.

Let  $Y$  be in  $G$ . Then there is some permutation matrix  $P$  such that  $v(\text{id})YP = v(\text{id})$ . By Lemma 5.4,  $YP$  is either the identity or  $YP = \frac{2}{d}J - U$ , where  $U$  and  $J$  are as in the statement of Lemma 5.4. Notice that in the second case  $YPU = \frac{2}{d}J - \mathbf{I}_d$ . Since  $P$  and  $U$  are permutation matrices, we conclude that  $G$  is indeed the product of the group of permutation matrices and the group  $\{\mathbf{I}_d, \frac{2}{d}J - \mathbf{I}_d\}$ . Since  $J$  commutes with the collection of permutation matrices, we obtain the direct product decomposition for  $G$  given in the statement of the proposition. ■

**Proof of Theorem 1.3** Let  $\mathcal{S} = \{n \mid \mathbb{Z}[q^n] = \mathbb{Z}[r^n]\}$ . From Theorem 2.1, we have  $\text{disc}(\mathbb{Z}[q^n]) = \text{disc}(\mathbb{Z}[r^n])$  for all  $n \in \mathcal{S}$ .

Let  $A_n$  denote the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $q_{i-1}^{n(j-1)}$  and let  $B_n$  be the  $d \times d$  matrix whose  $(i, j)$ -th entry is  $r_{i-1}^{n(j-1)}$ . Then  $\det(A_n)^2 = \text{disc}(\mathbb{Z}[q^n])$  and  $\det(B_n)^2 = \text{disc}(\mathbb{Z}[r^n])$ . Hence  $\det(A_n) = \pm \det(B_n)$  for all  $n \in \mathcal{S}$ . We have  $F(x) := \sum_{i=0}^{\infty} \det(A_i)x^i$  and  $G(x) := \sum_{i=0}^{\infty} \det(B_i)x^i$  are rational power series whose coefficients agree (up to sign) on some infinite set. By Theorem 4.1, they must agree (up to sign) on some arithmetic progression. Hence there exist  $a, b > 0$  and  $\epsilon \in \{-1, 1\}$  such that  $\det(A_{am+b}) = \epsilon \det(B_{am+b})$  for all  $m$  sufficiently large.

Now

$$\sum_{m=0}^{\infty} \det(A_{am+b})x^m = \epsilon \sum_{m=0}^{\infty} \det(B_{am+b})x^m$$

are rational functions. Hence they must have exactly the same poles. Let  $a_\sigma = \prod_{j=0}^{d-1} q_j^{-a\sigma(j)}$ . By Proposition 3.2 and Lemma 4.3, the  $a_\sigma$  are distinct and each must be a pole for each of these power series; *i.e.*, there can be no cancellation. Similarly,  $b_\sigma = \prod_{j=0}^{d-1} r_j^{-a\sigma(j)}$  has the property that  $b_\sigma$  is a pole of

$$\sum \det(B_{am+b})x^m = \epsilon \sum \det(A_{am+b})x^m$$

for all  $\sigma \in S_d$ . Since the two power series in the line above are the same up to sign we have that for each permutation  $\sigma$  there is some permutation  $\tau$  such that  $X(\sigma) := \prod_i q_i^{a\sigma(i)}$  is equal to  $Y(\tau) := \prod_i r_i^{a\tau(i)}$ . Moreover, the correspondence  $\sigma \mapsto \tau$  is a set-bijection of  $S_d$ . By Lemma 5.2, there exists some matrix  $E$  such that

$$(\sigma(0), \dots, \sigma(d-1))E = (\tau(0), \dots, \tau(d-1))$$

for all pairs  $(\sigma, \tau)$  such that  $X(\sigma) = Y(\tau)$ . By relabeling the conjugates of  $r$  so that  $X(\text{id})$  corresponds to  $Y(\text{id})$ , we may assume by Proposition 5.5 that  $E$  is either the identity or the matrix  $\frac{2}{d}J - U$ , where  $U$  is the permutation matrix whose  $(i, j)$  entry is  $\delta_{i+j, d+1}$ . Thus we are left with two cases.

Case I.  $E = \frac{2}{d}J - U$ .

In this case, we have  $v((i, i + 1))E = v((d - 2 - i, d - 1 - i))$  by equation (5.2).  
Hence

$$(v((i, i + 1)) - v(\text{id}))E = v((d - 2 - i, d - 1 - i)) - v(\text{id}).$$

Equivalently, for each  $i < d - 1$ ,  $q_i/q_{i+1} = r_{d-2-i}/r_{d-1-i} = r'_i/r'_{i+1}$ , where  $r'_i = r_{d-1-i}^{-1}$ . Thus  $q_0/q_j = r'_0/r'_j$  for  $0 \leq j \leq d - 1$ . Take  $s = q_0/r'_0 = q_0r_{d-1}$ . Then  $sr'_j = q_j$  for all  $j$ . Notice that

$$s^m((r'_0)^m + \cdots + (r'_{d-1})^m) = (q_0^m + \cdots + q_{d-1}^m).$$

Since  $(r'_0)^m + \cdots + (r'_{d-1})^m$  is a symmetric function of the  $r_i$ , it is rational; moreover, it is nonzero for infinitely many  $m$ . Similarly,  $q_0^m + \cdots + q_{d-1}^m$  is rational for all  $m$ . It follows that  $s^m$  is rational for some  $m$ . Since it is an algebraic integer, we conclude that it is an integer. Hence there are integers  $m, N$  with  $m > 0$  such that  $q^m r_{d-1}^m = N$ . Taking norms of both sides of this equation, we see that  $N^d = \text{Norm}(q)^m \text{Norm}(r)^m$ . The fact that  $X(\text{id}) = Y(\text{id})$  gives that  $q$  and  $r$  have the same norm up to sign. Thus  $q = \text{Norm}(r)^{2/d} \omega / r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ . This completes the proof in this case.

Case II.  $E$  is the identity matrix.

Assume that  $E$  is the identity matrix. Then since  $v((i, i + 1)) - v(\text{id}) = e_{i+1} - e_{i+2}$  is fixed by  $E$ , we see that  $q_i^a/q_{i+1}^a = r_i^a/r_{i+1}^a$  for all  $i$ . Let  $z = q_0^a/r_0^a$ . Then  $q_i^a/r_i^a = z$  for all  $i$ . Since  $v(\text{id})$  is fixed by  $E$ , we have

$$z^{\binom{d}{2}} \prod_i r_i^{ai} = \prod_i (zr_i)^{ai} = \prod_i q_i^{ai} = \prod_i r_i^{ai}.$$

Hence  $z^{\binom{d}{2}} = 1$  and so  $q_0/r_0$  is a root of unity. It follows that  $q = \omega r'$  for some root of unity  $\omega$  and some conjugate  $r'$  of  $r$ .

In either case we have that  $q = \omega r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ . This completes the proof. ■

**Proof of Corollary 1.4** By Theorem 1.3, we see that there are two cases to consider.

Case I.  $q = \omega r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ .

In this case  $|q| = |r'|$ . Since all conjugates of  $r$  are less than 1, we deduce that  $r' = r$ . Thus  $q = \omega r$ . Since  $q$  and  $r$  are both positive real numbers, we see that  $\omega = 1$  and hence  $q = r$ .

Case II.  $q = \text{Norm}(r)^{2/d} \omega / r'$  for some conjugate  $r'$  of  $r$  and some root of unity  $\omega$ .

We see that if  $\deg(q) = 2$ , then  $\text{Norm}(r)^{2/d} \omega / r' = \omega r''$  where  $r''$  is the conjugate of  $r'$ . Thus, for degree 2, it suffices to consider Case I only. So assume that  $\deg(r) \geq 3$ .

Let  $q = q_0, \dots, q_{d-1}$  denote the conjugates of  $q$  and let  $r_0, r_1, \dots, r_{d-1}$  denote the conjugates of  $r$ . By relabeling if necessary, we may assume that  $q_i = \text{Norm}(r)^{2/d} \omega_i / r_i$ , where  $\omega_i$  is a root of unity. Since  $d \geq 3$ , we can pick  $j > 0$  such that  $r_j \neq r$ . Then  $|q_j| = |\text{Norm}(r)^{2/d} / |r_j||$ . But this gives an immediate contradiction since  $|q_j|, |r_j| < 1$  and  $|\text{Norm}(r)| \geq 1$ . ■

We note that both possibilities given in the conclusion of Theorem 1.3 can occur. We give the following example to show this.

*Example.* Let  $q$  be a full rank algebraic integer of norm  $\pm 1$ . Then

$$\mathbb{Z}[q^n] = \mathbb{Z}[1/q^n] = \mathbb{Z}[(-q)^n]$$

for all integers  $n \neq 0$ .

**Proof** This follows from the fact that  $q^n$  has norm  $\pm 1$ . ■

## 6 Pisot Numbers of Small Degree

Here we prove our finiteness results for Pisot numbers of degree at most 3.

**Proof of Theorem 1.5** First suppose that  $\deg(r) = 2$  and let  $r'$  denote the conjugate of  $r$ . Let  $q$  be a Pisot number with  $\mathbb{Z}[q] = \mathbb{Z}[r]$  and let  $q'$  denote the conjugate of  $q$ . Then we can write  $q = br + a$ . Now  $\text{disc}(q) = b^2 \text{disc}(r)$  and hence  $b = \pm 1$ . Thus either  $q = a+r$  or  $q = a-r$ . If  $q = a+r$ , then  $q' = a+r'$  and hence  $|a+r'| < 1$ . There are at most two integers  $a$  which satisfy this inequality and so there are at most two Pisot numbers  $q$  of the form  $a+r$ . If, on the other hand,  $q = a-r$ , then  $q' = a-r'$  and so  $|a-r'| < 1$ . Again, we see that there are at most two integers  $a$  which give rise to a Pisot number  $q$  of the form  $a-r$ . Hence there are at most four Pisot numbers  $q$  such that  $\mathbb{Z}[r] = \mathbb{Z}[q]$ .

Next suppose that  $\deg(r) = 3$  and let  $r = r_0, r_1, r_2$  denote the conjugates of  $r$ . Let  $q$  be a Pisot number with  $\mathbb{Z}[q] = \mathbb{Z}[r]$ . Write  $q = a + br + cr^2$ . Then the conjugates of  $q$  are given by  $a + br_i + cr_i^2$  for  $i = 0, 1, 2$ . Hence

$$\begin{aligned} \text{disc}(q) &= \prod_{0 \leq i < j \leq 2} (b(r_i - r_j) + c(r_i^2 - r_j^2))^2 \\ &= \prod_{0 \leq i < j \leq 2} (r_i - r_j)^2 (b + c(r_i + r_j))^2 \\ &= \text{disc}(r)(b + c(r_0 + r_1))^2 (b + c(r_0 + r_2))^2 (b + c(r_1 + r_2))^2. \end{aligned}$$

Since  $\text{disc}(q) = \text{disc}(r)$ , we deduce that

$$(b + c(r_0 + r_1))^2 (b + c(r_0 + r_2))^2 (b + c(r_1 + r_2))^2 = 1.$$

Consider the polynomial in indeterminates  $x$  and  $y$

$$(6.1) \quad P(x, y) := (x + y(r_0 + r_1))(x + y(r_0 + r_2))(x + y(r_1 + r_2)).$$

Since  $P$  is a symmetric function of  $r_0, r_1, r_2$ , it is a homogeneous polynomial in  $x$  and  $y$  of degree 3 with integer coefficients. By the remarks above, we have  $P(b, c) = \pm 1$ . The polynomial  $P(x, y)$  can factor into at most three irreducible polynomials in  $\mathbb{Q}[x, y]$ .

If  $P(x, y)$  is not irreducible in  $\mathbb{Q}[x, y]$ , then it has at least one linear factor with rational coefficients. From equation (6.1) we then see that  $r_i + r_j$  must be rational for

some  $i \neq j$ . But  $r_0 + r_1 + r_2$  is also rational and hence  $r_k$  is rational for some  $k$ . But this contradicts the fact that  $r$  has degree 3, and so  $P(x, y)$  must be irreducible.

Hence the equation  $P(x, y) = \pm 1$  is a Thue equation. It is known that in the case of Thue equations, there are only a finite number of integer solutions to  $P(x, y) = \pm 1$  [8]. Moreover, for cubic Thue equations, there are at most 20 integer solutions to  $P(x, y) = \pm 1$  [2]. Thus there are at most 20 integer points  $(b, c)$  which satisfy  $P(b, c) = \pm 1$ . We claim that for particular  $b$  and  $c$  there are at most two  $a$  such that  $q = a + br + cr^2$  is a Pisot number. The reason for this is that the conjugates of  $q$  must be less than 1; that is,

$$|a + br_1 + cr_1^2| < 1 \quad \text{and} \quad |a + br_2 + cr_2^2| < 1.$$

Let  $\alpha = br_1 + cr_1^2$  and let  $\beta = br_2 + cr_2^2$ . Then there are at most two values of  $a$  such that  $|a + \alpha| < 1$  and so there are at most two values of  $a$  giving solutions to both equations. Since there are at most 20 values of  $(b, c)$  which give rise to Pisot numbers, there are at most 40 Pisot numbers  $q$  such that  $\mathbb{Z}[r] = \mathbb{Z}[q]$ . ■

In general, there are finitely many integer solutions  $(x, y)$  to a Thue equation  $P(x, y) = \pm k$ .

## 7 Pisot-Cyclotomic Numbers

In this section we prove that there are only finitely many Pisot-cyclotomic numbers of any fixed order. In fact we prove a stronger result. For this we use the Schmidt Subspace Theorem [7, Chapter VI].

**Theorem 7.1 (Schmidt Subspace Theorem)** *Let  $C$  be a positive constant and let  $\varepsilon > 0$ . If  $L_1, \dots, L_n$  are  $n$  independent linear homogeneous functions of  $\mathbf{x} = (x_1, \dots, x_n)$  with algebraic integer coefficients, then the set of points  $\mathbf{x} \in \mathbb{Z}^n$  such that*

$$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| < C \|\mathbf{x}\|^{-\varepsilon}$$

*is finite.*

We start with a Lemma.

**Lemma 7.2** *Let  $r_0, \dots, r_{d-1}$  be distinct nonzero complex numbers. Then the  $d - 1$  linear homogeneous forms*

$$L_i(\mathbf{x}) = (r_i - r_0)x_1 + (r_i^2 - r_0^2)x_2 + \cdots + (r_i^{d-1} - r_0^{d-1})x_{d-1}, \quad 1 \leq i \leq d - 1,$$

*are linearly independent over  $\mathbb{C}$ .*

**Proof** Let  $H_0(\mathbf{x}) = r_0x_1 + r_0^2x_2 + \cdots + r_0^{d-1}x_{d-1}$  and for  $1 \leq i \leq d - 1$ , define  $H_i(\mathbf{x}) = H_0(\mathbf{x}) + L_i(\mathbf{x})$ . It is sufficient to show that  $H_1, \dots, H_{d-1}$  are linearly independent

over  $\mathbb{C}$ . Suppose that some linear combination of them is zero, say  $\sum w_i H_i(\mathbf{x}) = 0$ . Then we can express this as a vector equation

$$\mathbf{w}^T V \mathbf{x} = [w_0 r_0, \dots, w_{d-1} r_{d-1}] \begin{bmatrix} 1 & r_0 & \cdots & r_0^{d-1} \\ 1 & r_1 & \cdots & r_1^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & \cdots & r_{d-1}^{d-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{bmatrix} = \mathbf{0},$$

where  $V$  is a Vandermonde matrix and  $\mathbf{w}$  is a nonzero column vector. Since this equation holds for all column vectors  $\mathbf{x}$ , we conclude that  $\mathbf{w}^T V = \mathbf{0}$ , contradicting the fact that  $V$  is invertible. The result follows. ■

**Lemma 7.3** *Let  $r$  and  $q$  be a Pisot numbers of degree  $d$  such that there exist integers  $c_0, \dots, c_{d-1}$  with the property that  $q = c_0 + c_1 r + \dots + c_{d-1} r^{d-1}$ . Then there exist positive constants  $C_1$  and  $C_2$ , dependent only on  $r$  (and hence independent of  $q$ ) such that*

$$C_1 q < \sqrt{c_0^2 + \dots + c_{d-1}^2} < C_2 q.$$

**Proof** To get the lower bound we use the Cauchy-Schwarz inequality, noting that

$$|q| = (c_0, \dots, c_{d-1}) \cdot (1, r, \dots, r^{d-1}) \leq \sqrt{c_0^2 + \dots + c_{d-1}^2} \sqrt{1^2 + r^2 + \dots + r^{2d-2}}.$$

Let  $\mathbf{c} = [c_0, \dots, c_{d-1}]^T$ . The upper bound relies on the fact that  $q$  and  $r$  are Pisot. Let  $q = q_0, q_1, \dots, q_{d-1}$  denote the conjugates of  $q$  and let  $\mathbf{q} = [q_0, q_1, \dots, q_{d-1}]^T$ . Then there is a Vandermonde matrix  $V$  whose entries are powers of conjugates of  $r$  such that

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{d-1} \end{bmatrix} = \begin{bmatrix} 1 & r_0 & r_0^2 & \cdots & r_0^{d-1} \\ 1 & r_1 & r_1^2 & \cdots & r_1^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = V \mathbf{c}.$$

Thus  $\|\mathbf{c}\| \leq \|V^{-1}\| \cdot \|\mathbf{q}\|$ . Note that  $\|V^{-1}\|$  depends only on  $r$  and since  $q$  is Pisot we have

$$\|\mathbf{q}\| \leq \sqrt{q^2 + |q_1|^2 + \dots + |q_{d-1}|^2} \leq \sqrt{q^2 + q^2 + \dots + q^2} \leq q\sqrt{d}.$$

Taking  $C_2 = \|V^{-1}\| \sqrt{d}$  completes the proof. ■

**Proof of Theorem 1.6** We can, without loss of generality, assume that  $r$  is a Pisot number. For take  $r'$  a Pisot number such that  $\mathbb{Z}[r] = \mathbb{Z}[r']$ . If no such  $r'$  exists then there are no (hence finitely many) Pisot numbers  $q$  such that  $\mathbb{Z}[r] = \mathbb{Z}[q]$ . If such an  $r'$  exists,  $\mathbb{Z}[q] = \mathbb{Z}[r]$  if and only if  $\mathbb{Z}[q] = \mathbb{Z}[r']$ .

Let  $q$  be a Pisot number satisfying  $\mathbb{Z}[q] = \mathbb{Z}[r]$ . Let  $q = q_0, \dots, q_{d-1}$  denote the conjugates of  $q$ , and similarly  $r = r_0, \dots, r_{d-1}$  denote the conjugates of  $r$ . Then there exists an integer polynomial  $P(x) = c_0 + \dots + c_{d-1} x^{d-1}$  of degree at most  $d-1$  such that for some labeling  $r_0, r_1, \dots, r_{d-1}$  we have

$$(7.1) \quad q_i = P(r_i) = c_0 + c_1 r_i + \dots + c_{d-1} r_i^{d-1}$$

and  $q_0, r_0 > 1$ . Further, we have

$$\text{disc}(r) = \text{disc}(q) = \prod_{i < j} (q_i - q_j)^2$$

is fixed, for all  $q$ . Using equation (7.1), we see that

$$q_i - q_j = P_{i,j} := c_1(r_i - r_j) + \cdots + c_{d-1}(r_i^{d-1} - r_j^{d-1}).$$

Consider the  $d(d-1)$  linear homogeneous polynomials

$$L_{i,j}(\mathbf{x}) := (r_i - r_j)x_1 + \cdots + (r_i^{d-1} - r_j^{d-1})x_{d-1} \quad 1 \leq i, j \leq d, i \neq j$$

and let  $\mathbf{c} = (c_1, \dots, c_{d-1})$ . Notice that

$$\begin{aligned} q_i - q_j &= P(r_i) - P(r_j) = c_0 + c_1 r_i + \cdots + c_{d-1} r_i^{d-1} - (c_0 + c_1 r_j + \cdots + c_{d-1} r_j^{d-1}) \\ &= c_1(r_i - r_j) + \cdots + c_{d-1}(r_i^{d-1} - r_j^{d-1}) = L_{i,j}(\mathbf{c}). \end{aligned}$$

Then  $|\text{disc}(q)| = \prod_{i < j} |L_{i,j}(\mathbf{c})|^2$ . For  $0 \leq i \leq d-1$ , define

$$Q_i(\mathbf{x}) = \prod_{j \leq d-1, j \neq 0, i} L_{i,j}(\mathbf{x}).$$

Then

$$|\text{disc}(q)| = |Q_1(\mathbf{c}) \cdots Q_{d-1}(\mathbf{c})(q_0 - q_2)^2 \cdots (q_0 - q_d)^2|.$$

By assumption  $q_1, \dots, q_{d-1}$  are in the unit disc, and so we have

$$|(q_0 - q_2)^2 \cdots (q_0 - q_{d-1})^2| > (q - 1)^{2d-2}.$$

From this we see that  $|Q_1(\mathbf{c}) \cdots Q_{d-1}(\mathbf{c})(q - 1)^{2d-2} < |\text{disc}(q)|$  and thus  $|Q_i(\mathbf{c})| < (q - 1)^{-2} |\text{disc}(q)|^{1/(d-1)}$  for some  $i$ . We are almost ready to apply the subspace theorem to the polynomials  $L_{i,0}, \dots, L_{i,d-1}$  (with  $L_{i,i}$  omitted). Observe that these homogeneous linear forms are linearly independent over  $\mathbb{C}$  by Lemma 7.2. Next observe that if  $\mathbf{x} \in \mathbb{Z}^d$ , then

$$|L_{i,0}(\mathbf{x}) \cdots L_{i,d-1}(\mathbf{x})| = |Q_i(\mathbf{x})| |L_{i,0}(\mathbf{x})|.$$

We have

$$\begin{aligned} |L_{i,0}(\mathbf{x})| &= |(r_i - r_0)x_1 + \cdots + (r_i^{d-1} - r_0^{d-1})x_{d-1}| \\ &\leq \| [r_i - r_0, \dots, r_i^{d-1} - r_0^{d-1}] \| \cdot \| [x_1, x_2, \dots, x_{d-1}] \| \\ &\leq \| [r + 1, \dots, r^{d-1} + 1] \| \cdot \| \mathbf{x} \| \\ &\leq \| [r^{d-1} + 1, \dots, r^{d-1} + 1] \| \cdot \| \mathbf{x} \| \\ &\leq \sqrt{d-1} \cdot |r^{d-1} + 1| \cdot \| \mathbf{x} \|. \end{aligned}$$

Let  $0 < \varepsilon < 1$ . By Schmidt's subspace theorem, for any positive  $C$ , there are only finitely many  $\mathbf{x} \in \mathbb{Z}^{d-1}$  such that  $|L_{i,0}(\mathbf{x}) \cdots L_{i,d-1}(\mathbf{x})| < C\|\mathbf{x}\|^{-\varepsilon}$ . Consequently, there are only finitely many integer points  $\mathbf{x}$  such that  $|Q_i(\mathbf{x})| < \|\mathbf{x}\|^{-1-\varepsilon}$ , since for each such point  $\mathbf{x}$ , we have

$$\begin{aligned} |L_{i,0}(\mathbf{x}) \cdots L_{i,d-1}(\mathbf{x})| &= |Q_i(\mathbf{x})| |L_{i,0}(\mathbf{x})| < \|\mathbf{x}\|^{-1-\varepsilon} \sqrt{d-1} \cdot |r^{d-1} + 1| \cdot \|\mathbf{x}\| \\ &= C\|\mathbf{x}\|^{-\varepsilon}, \end{aligned}$$

for some constant  $C$ .

By Lemma 7.3, there exist positive constants  $C_0$  and  $C_1$  which depend only on  $r$  such that  $C_0q \leq \|\mathbf{c}\| \leq C_1q$ .

Thus there are only finitely many Pisot numbers  $q$  (with corresponding integer vectors  $\mathbf{c}$ ) such that

$$\|Q_i(\mathbf{c})\| < \|\mathbf{c}\|^{-1-\varepsilon} \leq (C_0q)^{-1-\varepsilon} \leq C_0^{-1-\varepsilon} \cdot q^{-1-\varepsilon} \leq C_2 \cdot q^{-1-\varepsilon},$$

(here  $C_2 = C_0^{-1-\varepsilon}$ ). Suppose that there are infinitely many Pisot numbers  $q$  with  $\mathbb{Z}[q] = \mathbb{Z}[r]$ . Then we have just shown that  $|Q_i(\mathbf{c})| \geq C_2q^{-1-\varepsilon}$  for infinitely many such  $q$ . But we know  $|Q_i(\mathbf{c})| < (q-1)^{-2} |\text{disc}(q)|^{1/(d-1)} = (q-1)^{-2} \text{disc}(r)^{1/(d-1)}$ , and so we have  $C_2q^{-1-\varepsilon} < (q-1)^{-2} |\text{disc}(q)|^{1/(d-1)}$  for infinitely many Pisot numbers  $q$  with  $\mathbb{Z}[q] = \mathbb{Z}[r]$ . Equivalently,

$$\frac{(q-1)^2}{q^{1+\varepsilon}} < \frac{\text{disc}(r)^{1/(d-1)}}{C_3}$$

for infinitely many Pisot numbers  $q$  with  $\mathbb{Z}[q] = \mathbb{Z}[r]$ . But we see then that there is a computable upper bound for  $q$ , as  $\frac{(q-1)^2}{q^{1+\varepsilon}} \rightarrow +\infty$  as  $q \rightarrow +\infty$ . But Pisot numbers in a number field are discrete, and so we obtain a contradiction. The result now follows. ■

## 8 Conclusions, Open Questions, and Conjectures

A number of finiteness results are shown in this paper. Unfortunately, Theorem 1.5 only provides a bound for the number of Pisot numbers for degrees 2 and 3. In addition, these bounds are probably not best possible. Theorem 1.6 proves that there are a finite number of Pisot numbers  $q$  such that  $\mathbb{Z}[q] = \mathbb{Z}[r]$ , given some nice restrictions on  $r$ . Unfortunately, no upper bounds are given on the number of Pisot numbers  $q$  of this form. We therefore state the following open problems.

- (1) Improve the bounds given in the statement of Theorem 1.5.
- (2) Improve Theorem 1.6 to give bounds for the number of Pisot numbers  $q$  such that  $\mathbb{Z}[q] = \mathbb{Z}[r]$  in terms of  $r$ .
- (3) Extend the results of Theorem 1.6 to say something about the case where the conjugates of  $r$  do not all lie in  $\mathbb{Q}[r]$ , or give an example to show that the finiteness property does not hold.

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