

GENERALIZED GORSHKOV-WIRSING POLYNOMIALS AND THE INTEGER CHEBYSHEV PROBLEM

KEVIN G. HARE

ABSTRACT. The *Integer Chebyshev Problem* is the problem of finding an integer polynomial of degree n such that the supremum norm on $[0, 1]$ is minimized. The most common technique used to find upper bounds is by explicit construction of an example. This is often (although not always) done by heavy computational use of the LLL and the Simplex method. Among the first methods developed to find lower bounds was through a sequence of polynomials known as the Gorshkov-Wirsing polynomials.

This paper studies properties of the Gorshkov-Wirsing polynomials. It is shown how to construct generalized Gorshkov-Wirsing polynomials on any interval $[a, b]$, with $a, b \in \mathbb{Q}$. An extensive search for generalized Gorshkov-Wirsing polynomials is done for a large family of $[a, b]$. Using generalized Gorshkov-Wirsing polynomials, LLL and the Simplex method, upper and lower bounds for the Integer Chebyshev Constant on intervals other than $[0, 1]$ are calculated. These methods are compared with other existing methods.

1. INTRODUCTION

We define the supremum norm on an interval $I = [a, b]$ as:

$$\|p\|_I := \sup_{z \in I} |p(z)|.$$

For the purposes of this paper, we assume that I is a rational interval on the real line, but the Integer Chebyshev Problem can be extended to any compact set of the complex numbers. The case of finding monic polynomials with real coefficients and minimal supremum norm on I is related to the logarithmic capacity of the set I , and leads to the study of Chebyshev polynomials [12, 25]. The Integer Chebyshev Problem is the problem of finding an integer polynomial of degree n with minimal supremum norm on the interval $[0, 1]$. This supremum is normalized by taking the n^{th} root. In more general terms, we define this value on any interval $[a, b]$ as:

$$t_{\mathbb{Z},n}[a, b] = \inf \left\{ \|p(x)\|_{[a,b]}^{1/n} \mid p(x) \in \mathbb{Z}[x], \deg(p(x)) \leq n, p(x) \neq 0 \right\}.$$

By noticing that

$$t_{\mathbb{Z},n+m}[a, b]^{n+m} \leq t_{\mathbb{Z},n}[a, b]^n t_{\mathbb{Z},m}[a, b]^m$$

we see that the limit:

$$t_{\mathbb{Z}}[a, b] = \lim_{n \rightarrow \infty} t_{\mathbb{Z},n}[a, b]$$

is well defined.

The first few known values of the Integer Chebyshev Problem on the interval $[0, 1]$ are given in Tables 1.

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Habsieger and Salvy [13] determine a complete list of polynomials up to and including degree 75. This was extended by Meichsner to degree 230 [21]. For a good survey of results concerning the Integer Chebyshev Problem, see [3, 4, 24].

Borwein and Erdelyi [5] show that $t_{\mathbb{Z}}[0, x]$ is continuous and constant near $x = 1$. They also showed that $\left(m + 2 - \frac{1}{4(m+1)}\right)^{-1} \leq t_{\mathbb{Z}}[0, 1/m]$. A study of the intervals $[r/s, r/s + \delta]$ as $\delta \rightarrow 0$ is done by Flammang, Rhin and Smyth [9]. Chudnovsky and Ferguson looked at the multidimensional case [6, 7]. The study of the Integer Chebyshev Problem on a compact set in \mathbb{C} , and its relationship to Mahler measure is done in [11]. The study of the related case, that of minimal monic integer polynomials, was pioneered by Borwein, Pinner and Pritsker [2]. This work has been followed up by [16, 17, 18].

Bounds have been given on the frequency of certain factors for large degree Integer Chebyshev polynomials. For example, Pritsker shows that $x(1-x)$ must show up as a factor between 0.2961 and 0.3634 of the time, (normalized by the degree) [23].

Observe that:

$$t_{\mathbb{Z}}[a, b] \leq t_{\mathbb{Z}_n}[a, b].$$

Thus, techniques to find an upper bound for $t_{\mathbb{Z}}[a, b]$ center around finding good examples of $t_{\mathbb{Z}_n}[a, b]$ for large values of n . For example Borwein and Erdelyi [5] show:

$$t_{\mathbb{Z}}[0, 1] \leq \frac{1}{2.3543\dots}$$

by considering a degree 210 polynomial. In the same paper, Borwein and Erdelyi show how this bound could be improved to $\frac{1}{2.3605\dots}$. This is further refined by Habsieger and Salvy [13] to get:

$$t_{\mathbb{Z}}[0, 1] \leq \frac{1}{2.3612\dots}$$

Recently Pritsker [24], by use of weighted potential theory has improved both the lower and upper bounds to get

$$\frac{1}{2.3736} \leq t_{\mathbb{Z}}([0, 1]) \leq \frac{1}{2.3629}$$

This was again refined by Flammang [8] to give

$$t_{\mathbb{Z}}[0, 1] \leq \frac{1}{2.3646\dots}$$

The method that Borwein and Erdelyi, and later Habsieger and Salvy use to find these bounds is to consider a set of polynomials $p_i(x)$ that are factors of a large Integer Chebyshev polynomial, and consider the problem of minimizing ℓ where $\sum r_i \log(|p_i(x)|) / \deg(p_i) \leq \ell$, for all $x \in [a, b]$ with $\sum r_i = 1$, $0 \leq r_i$. This is the logarithm of an Integer Chebyshev Problem. By choosing a large number of points $x \in [0, 1]$, instead of the entire interval, they get a system of linear equations, on which the Simplex method can be used to get a good estimate [5, 13, 26]. The initial set of polynomials p_i can be found by using LLL on the basis $1, x, x^2, \dots, x^n$, and using the inner product of $\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)dx$ [15, 19]. Other techniques are discussed later, in Section 2. The refinement to this upper bound by Flammang was done in much the same way, but instead of discretizing the linear programming problem, she used a method of semi-infinite programming introduced by Smyth

[8, 27]. All of these methods are discussed in Section 2, improvements to the simplex method are discussed there as well.

Numerous methods have been proposed for lower bounds as well. The methods that we will focus on in this paper is that of Gorshkov-Wirsing polynomials. Other methods (and their limitations) are discussed in Section 3.

The following result, which is a simple consequence of properties of resultants, is needed before we begin our discussion of Gorshkov-Wirsing polynomials. For more discussion see [4].

Lemma 1.1. *Suppose $q(x) \in \mathbb{Z}[x]$ and $\deg(q(x)) = n$, and suppose that $p(x) = a_k x^k + \dots + a_0 \in \mathbb{Z}[x]$, $a_k > 0$ has all of its roots in the interval $[a, b]$. If $\gcd(p(x), q(x)) = 1$ then:*

$$(\|q(x)\|_{[a,b]})^{1/n} \geq a_k^{-1/k}.$$

This lemma says that, if $p(x)$ has all roots in I , and its lead coefficient is relatively small, then we will need $p(x)$ as a factor for large degree Integer Chebyshev polynomials. We formalize this by saying such $p(x)$ are critical.

Definition 1.1. We say an irreducible polynomial $p(x) = a_k x^k + \dots + a_0$ is critical for a interval I if

- All roots of p are in the interval I , and
- $|a_k|^{-1/k} \geq t_{\mathbb{Z}}(I)$.

Notice that, in the calculation of $t_{\mathbb{Z}_n}(I)$, we can only have a finite number of critical polynomials. This gives us a simple corollary to Lemma 1.1 which is the basis of the study of Gorshkov-Wirsing polynomials.

Corollary 1.1. *If there exists an infinite family p_1, p_2, \dots , such that all roots of $p_i(x) = a_{n_i} x^{n_i} + \dots + a_0$ are in an interval I , then*

$$t_{\mathbb{Z}}(I) \geq \liminf_{i \rightarrow \infty} |a_{n_i}|^{-1/n_i}$$

With Corollary 1.1 in mind, we can now discuss how one would find such an infinite family of polynomials. Denote:

$$U(x) := \frac{x(1-x)}{1-3x(1-x)},$$

and further denote:

$$p_0(x) := 2x - 1.$$

Define the sequence of polynomials $p_i(x)$ recursively by:

$$\begin{aligned} p_i(x) &:= \text{numer}(p_{i-1}(U(x))) \\ &= (1 - 3x + 3x^2)^{\deg(p_{i-1})} p_{i-1}(U(x)). \end{aligned}$$

normalized to have integer coefficients, no integer content, and positive lead coefficient. The first few polynomials in this sequence are:

$$\begin{aligned} p_0(x) &= 2x - 1, \\ p_1(x) &= 5x^2 - 5x + 1, \\ p_2(x) &= 29x^4 - 58x^3 + 40x^2 - 11x + 1, \\ p_3(x) &= 941x^8 - 3764x^7 + 6349x^6 - 5873x^5 + 3243x^4 - 1089x^3 + 216x^2 - 23x + 1. \end{aligned}$$

This defines an infinite sequence of polynomials $p_i(x)$, known as the Gorshkov-Wirsing polynomials, [4, 20, 22], with lead coefficient a_{n_i} . This sequence has the nice properties that

- All roots are in $[0, 1]$,
- All polynomials are irreducible,
- Polynomials are of degree 2^i with lead coefficient a_{n_i} ,
- $\lim_{i \rightarrow \infty} |a_{n_i}|^{-1/2^i} = 1/2.3768417062\dots$

Combining these properties with Corollary 1.1 gives us the lower bound:

$$t_{\mathbb{Z}}[0, 1] \geq \frac{1}{2.3768417062}.$$

Borwein and Erdelyi [5] show that this bound is not tight, and that there exists an $\epsilon > 0$ such that

$$t_{\mathbb{Z}}[0, 1] \geq \frac{1}{2.3768417062} + \epsilon.$$

Their argument relies on the fact that the end points of the interval are the roots of critical polynomials on the interval.

In this paper we show how to generalize the definition of Gorshkov-Wirsing polynomials to give different sequences of polynomials, and derive different bounds for different intervals.

2. UPPER BOUND TECHNIQUES

In this section we give a review of some of the methods to find upper bounds for $t_{\mathbb{Z}}(I)$. The first two methods are that of Amoroso [1] and Habsieger and Salvy [13]. For proof of correctness, see the original articles. The next two methods involve LLL and the Simplex method, and use as a guiding principle that good example of a polynomial with small norm gives a good upper bound. These upper bounds are compared in Table 2, 4, 6 and 8. For comparison, the best lower bound known is also given.

2.1. Amoroso. For a more complete discussion of this technique, see [1].

Define

$$\begin{aligned} \rho(r_1, r_2) = & -\frac{1}{1-r_1-r_2} \left((r_1+r_2) \log(2) - \frac{(1-r_1-r_2)^2}{4} \log(1-r_1-r_2) \right. \\ & - \frac{(1+r_1-r_2)^2}{4} \log(1+r_1-r_2) - \frac{(1-r_1+r_2)^2}{4} \log(1-r_1+r_2) \\ & \left. - \frac{(1+r_1+r_2)^2}{4} \log(1+r_1+r_2) + r_1^2 \log(2r_1) + r_2^2 \log(2r_2) \right) \end{aligned}$$

and define

$$f^+(r_1, r_2) = \log \left(\sqrt{\frac{|I|}{4}} \right) + r_1 \log(b_1 \sqrt{\delta}) + r_2 \log(b_2 \sqrt{\delta}) + \rho(r_1, r_2).$$

Then

$$t_{\mathbb{Z}}(I) \leq \exp \min_T f^+(r_1, r_2),$$

where T is the simplex $0 \leq r_1, r_2$ and $r_1 + r_2 \leq 1$.

As can be seen, the upper bound attained is very much dependant on the denominators b_1 and b_2 . In fact, it is sometimes advantageous to consider an upper bound based on a slightly larger interval, along with the observation that if $I \subset J$

then $t_{\mathbb{Z}}(I) \leq t_{\mathbb{Z}}(J)$. Such an example can be seen by $I = [1/24, 1 - 1/24]$ and $J = [0, 1]$. Amoroso's lower bound estimate based on this I is then $\frac{1}{29.2213}$, where as on J is $\frac{1}{2.4141}$. This is taken into account when compiling the data in the Tables. Table 10 demonstrates what happens if this is not taken into account.

2.2. Habsieger and Salvy. The method of Habsieger and Salvy are used to explicitly find the best polynomial of lower degree. For degree n , we first find a reasonably good polynomial, using LLL, for example, and hence a reasonably good upper bound ℓ . We next use this bound for good polynomials to find required factors of the best polynomial of degree n . This is done by noticing that, for any polynomial P , we have from Markov's inequality that

$$\max_{a \leq x \leq b} |P^{(r)}(x)| \leq \frac{2^r}{(b-a)^r} \cdot \frac{n^2(n^2-1^2)(n^2-2^2) \cdots (n^2-(r-1)^2)}{(2r-1)!!} \max_{a \leq x \leq b} |P(x)|$$

where $(2i+1)!! = 1 \cdot 3 \cdot 5 \cdots (2i+1)$. Hence, if we consider an irreducible factor $p(x)$ with all its roots in an interval $[a, b]$, and consider the product of the left hand side over these roots, then the result is 0 if $P^{(r)}$ has $p(x)$ as a factor, or is greater than or equal to $1/a_k$, where a_k is the lead coefficient of $p(x)$. The right hand side is explicitly computable, based on the good upper bound ℓ , found by LLL. This gives us a bound for the right hand side of the equation, which when combined with the restrictions on the left hand side may imply that the left hand side of the equation is 0. Hence for some values of r , we see that the left hand side must then equal 0, which in turn implies a multiplicity of the factor $p(x)$.

Then, using the product of all required factors, say $Q(x)$, of degree k , and this good bound, and a random selection of points in the interval, say x_i , we notice that the best polynomial will satisfy

$$-\ell \leq (a_{n-k}x_i^{n-k} + \cdots + a_0)Q(x) \leq \ell$$

for all x_i , and further $a_{n-k} \geq 1$. This is a system of linear equations. We can solve for all integer solutions of the a_i 's exhaustively, and then select the best example(s) from this list.

One nice benefit is that factors of the best polynomials are useful to add to the basis of the Simplex method.

This is done up to degree 3, which although not large, is good enough for our purposes.

2.3. LLL. The use of LLL tends to give a very crude estimate of an upper bound. One side benefit of this method though is that it tends to give a very good set of polynomials for which to start the Simplex method.

Initially we consider a basis $1, x, \cdots, x^n$ and an inner product $\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)dx$. Small elements in this basis have small 2-norm, which tends to mean small sup-norm. So using LLL, we get an element with small norm, say $p_1(x)$. We then repeat this process with a basis $p_1(x), x \cdot p_1(x), \cdots, x^n \cdot p_1(x)$, and iterate the process.

This is done up to $n = 20$.

2.4. Simplex Method. Consider a set of polynomials $p_i(x)$ that are factors of a large Integer Chebyshev polynomial, and consider the problem of minimizing ℓ where $\sum r_i \log(|p_i(x)|) / \deg(p_i) \leq \ell$, for all $x \in [a, b]$ with $\sum r_i = 1$, $0 \leq r_i$. This is the logarithm of an Integer Chebyshev Problem. By choosing a large number of

points $x \in [0, 1]$, instead of the all of them, we get that this is a linear programming problem. Now, one area that can be influenced is by careful choice of the x_j . Initially we choose 50 points, in I uniformly distributed. We then consider the resulting object $\sum r_i \log(|p_i(x)|) / \deg(p_i)$ and find its local maxima. We add these local maxima to our set of x_j and iterate. This has the advantage that each iteration focusses more and more attention to the “problem” spots.

If the factors exist, then this is done up to using 20 factors. Factors were chosen such that

- All known critical polynomials were included.
- Good factors found through LLL, or Habsieger and Salvy were included.
- Factors that proved useful for smaller examples were included.
- Polynomials found with small $a_k^{-1/k}$ values, and all with roots in the interval were included. These were normally found through searches for Gorshkov-Wirsing pairs.

3. LOWER BOUND TECHNIQUES

There are many ways in which to estimate lower bounds on ICP. We will outline some of the methods below. For a correctness of the method, the reader is encourage to read the original article. These methods are summarized in Tables 3, 5, 7 and 9.

3.1. Amoroso. For a more complete discussion see [1].

Let $I = [a_1/b_1, a_2/b_2]$ be a rational interval. Let $\delta = a_2/b_2 - a_1/b_1$ be its diameter. Define

$$\begin{aligned} h(r_1, r_2) = & 1/2(1 - r_1 - r_2) \log(1 - r_1 - r_2) - 1/2(1 + r_1 + r_2) \log(1 + r_1 + r_2) \\ & - 1/2(1 + r_1 - r_2) \log(1 + r_1 - r_2) - 1/2(1 - r_1 - r_2) \log(1 - r_1 - r_2) \\ & + 2r_0 \log(2r_0) \end{aligned}$$

and define

$$f^-(r_1, r_2) = \log(\delta) + \max((r_1 - 1) \log(b_1 \delta) + h(r_1, r_2), (r_2 - 1) \log(b_2 \delta) + h(r_2, r_1))$$

Then

$$t_{\mathbb{Z}}(I) \geq \exp(\inf_T f^-(r_1, r_2))$$

As before, the size of the denominator plays a crucial role in this estimate, and often it is advantageous to consider a smaller interval with a small denominator.

3.2. Flammang, Rhin, Smyth. For a more complete description, see [9]. In this method, the authors restricted their attention to Farey intervals $[p/q, r/s]$. That is, intervals where $qr - ps = 1$. They define

$$\begin{aligned} U_0 &= z & V_0 &= 1 \\ U_{k+1} &= U_k^2 + V_k^2 & V_{k+1} &= U_k V_k \\ x_k &= U_k / V_k \end{aligned}$$

Further, take

$$g_-(z) = \prod x_k^{-1/2^k}$$

Then

$$t_{\mathbb{Z}}(I) \geq \frac{1}{\sqrt{qs}} g_-(\sqrt{q/s})$$

Using the fact that if $I \subset J$ then $t_{\mathbb{Z}}(I) \leq t_{\mathbb{Z}}(J)$, and the fact that every interval has a maximal Farey subinterval, we can extend this lower bound to all intervals. (Although, sometimes quite badly.) Table 11 demonstrates what happens if this is not taken into account.

3.3. Flammang. For a more complete description, see [10].

Again, focus is given to Farey Intervals Let $I = [p/q, r/s]$. Let $\lambda_0 = \frac{qs}{q+s}$ and $\lambda_{k+1} = \frac{\lambda_k}{(1+\lambda_k)^2}$ Then

$$t_{\mathbb{Z}}(I) \geq \frac{1}{q+s} \prod (1 + \lambda_i)^{-1/2^{i+1}}.$$

Same comments as before with respect to Farey Intervals.

4. GENERALIZED GORSHKOV-WIRSING RATIONAL FUNCTIONS

In Section 1 we considered the function

$$U(x) = \frac{x(1-x)}{1-3x(1-x)}.$$

The property of this function that makes it useful for finding lower bounds is that it is of degree 2, and it maps $[0, 1]$ to itself twice. Hence, if $p(x)$ is a polynomial with all of its roots in $[0, 1]$ then $\text{numer}(p(U(x)))$ has all of its roots in $[0, 1]$.

We extend this concept to give the definition:

Definition 4.1. A *generalized Gorshkov-Wirsing rational function* on $[a, b]$ is a rational function:

$$U(x) = \frac{r(x)}{s(x)}$$

which maps the interval $[a, b]$ to itself d times, where $\deg(b(x)) \leq \deg(t(x)) = d$, and $b(x), t(x) \in \mathbb{Z}[x]$. Denote the set of all such $U(x)$ by $\mathcal{U}[a, b]$.

We now give a complete description of $\mathcal{U}[a, b]$, if $a, b \in \mathbb{Q}$.

Theorem 4.1. Let $a, b \in \mathbb{Q}$, and $p(x)$ and $q(x)$ be integer polynomials, non-negative on $[a, b]$ with the following properties

- $\deg(p) = \deg(q)$
- Both $p(x)$ and $q(x)$ are totally real, that is, all of their roots are real.
- The polynomials $p(x)$ has s single root at a , and double roots at $\alpha_1 < \alpha_2 < \dots$. If $\deg(p)$ is even, then $p(x)$ also has a single root at b .
- The polynomials $q(x)$ has double roots at $\beta_1 < \beta_2 < \dots$. If $\deg(q)$ is odd, then $q(x)$ also has a single root at b .
- The roots interlace, that is

$$a < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \dots < b$$

Then, for all $e, f \in \mathbb{N}$

$$(1) \quad \frac{a \cdot e \cdot p(x) + b \cdot f \cdot q(x)}{e \cdot p(x) + f \cdot q(x)} \quad \text{and} \quad \frac{b \cdot e \cdot p(x) + a \cdot f \cdot q(x)}{e \cdot p(x) + f \cdot q(x)}$$

are all members of $\mathcal{U}[a, b]$. Furthermore, all $U(x) \in \mathcal{U}[a, b]$ can be written in this form.

Such pairs of polynomials p and q play an important role in this study, and we will denote them as Gorshkov-Wirsing pairs.

Examining equation (1) of Theorem 4.1 gives the examples:

$$\begin{aligned} \frac{1-5x^2}{6x^2-2} &= \frac{\frac{1}{2}(1-4x^2)-\frac{1}{2}x^2}{(1-4x^2)+x^2} \in \mathcal{U}[-1/2, 1/2], \\ \frac{x(2x-1)}{-2+10x-14x^2} &= \frac{\frac{1}{2}x(1-2x)+0(3x-1)^2}{(x(1-2x))+3(3x-1)^2} \in \mathcal{U}[0, 1/2], \\ 2x^2 - 1 &= \frac{x^2-(1-x^2)}{x^2+(1-x^2)} \in \mathcal{U}[-1, 1]. \end{aligned}$$

Proof of Theorem 4.1. Denote (1) by $U(x)$. We see that $U(x) = a$ between a and b precisely when $U(x) - a = 0$. Upon simplification we see that this is precisely when $p(x) = 0$. Similarly, we see that $U(x) = b$ precisely when $q(x) = 0$. As $p(x)$ and $q(x)$ are non-negative, and do not share any roots on the interval $[a, b]$, we have that $p(x) + q(x)$ is positive on $[a, b]$. By considering the degrees of $p(x)$ and $q(x)$ and the fact that the roots interlace, we get that $U(x)$ maps $[a, b]$ to itself d times, where $\deg(p) = \deg(q) = d$. Hence $U(x) \in \mathcal{U}[a, b]$.

To see that all $U(x)$ must be of this form, we consider two cases.

Case: 1 Assume that the degree of the numerator is odd, say $2n + 1$. We notice that either $U(a) = a$ and $U(b) = b$, or $U(a) = b$ and $U(b) = a$. By noticing that

$$\frac{bq(x) + ap(x)}{p(x) + q(x)} = b + a - \frac{aq(x) + bp(x)}{p(x) + q(x)}$$

it suffices to show the result for the first situation. So we can assume that $U(a) = a$ and $U(b) = b$. Notice that $U(x) - a$ has $n + 1$ roots, say $a < \alpha_1 < \alpha_2 < \dots < \alpha_n$. Further we see that $U(x) - a$ must have double roots at $\alpha_1 < \alpha_2 < \dots < \alpha_n$, else $U(x) < a$ for some $x \in [a, b]$. Write $p_1(x) = (x - a)(x - \alpha_1)^2 \dots (x - \alpha_n)^2$.

Similarly we see that $U(x) - b$ has $n + 1$ roots, that interlace with the α_i , and that all but b must be a double root. Call these roots $\beta_1 < \dots < \beta_n < b$. Denote $q_1(x) = (b - x)(x - \beta_1)^2 \dots (x - \beta_n)^2$.

Writing $U(x) = \frac{r(x)}{s(x)}$, we notice that

$$U(x) - a = \frac{e \cdot p_1(x)}{s(x)}$$

and further that:

$$U(x) - b = \frac{-f \cdot q_1(x)}{s(x)}$$

for $e, f > 0$. This gives that $(b - a)s(x) = e \cdot p_1(x) + f \cdot q_1(x)$.

Thus we can write:

$$U(x) = \frac{b \cdot e \cdot p_1(x) + a \cdot f \cdot q_1(x)}{e \cdot p_1(x) + f \cdot q_1(x)}$$

By multiplying the numerator and denominator by the appropriate integer, we may assume that e and f are integers.

Case: 2 The case when the degree of the numerator is even is similar. □

By simple algebra, we get the following lemma:

Lemma 4.1. *Let (P, Q) and (p_1, q_1) be Gorshkov-Wirsing pairs satisfying the conditions of Theorem 4.1. Let $e, f \in \mathbb{N}$. Define inductively the generalized Gorshkov-Wirsing rational function:*

$$U_i(x) = \frac{a \cdot e \cdot p_i(x) + b \cdot f \cdot q_i(x)}{e \cdot p_i(x) + f \cdot q_i(x)}.$$

$$p_{i+1}(x) = \text{numer}(P(U_i(x))),$$

$$q_{i+1}(x) = \text{numer}(Q(U_i(x))),$$

normalized so that p_{i+1} and q_{i+1} are integer polynomials with no integer content, and are positive on $[a, b]$. Then:

- p_i and q_i satisfy the conditions of Theorem 4.1.
- $\deg(p_i(x)) = \deg(q_i(x)) = \deg(p_1(x))^{\deg P(x)}$.

In practice we take e and f so that $e \mid \text{denom}(a)$ and $f \mid \text{denom}(b)$.

Consider for example $[a, b] = [0, 1]$, and take $p_1(x) = P(x) = (2x - 1)^2$ and $q_1(x) = Q(x) = x(1-x)$. Take $e = f = 1$. This gives us that $U(x) = \frac{x(1-x)}{x(1-x) + (2x-1)^2} = \frac{x(1-x)}{1-3x(1-x)}$, and $p_2(x) = (5x^2 - 5x + 1)^2$ and $q_2(x) = x(1-x)(2x-1)^2$. In fact, this is the start of the classic Gorshkov-Wirsing sequence (squared).

This leads to the definition:

Definition 4.2. A set of polynomials (P, Q) , (p_i, q_i) , and integers e, f that satisfy the conditions of Lemma 4.1 are called *generalized Gorshkov-Wirsing polynomials*.

It is worth noting that given two pairs of Gorshkov-Wirsing pairs that there are 8 different ways that they can be combined to give a Gorshkov-Wirsing sequence.

At this point we need some discussion on how to ensure p_{n+1} and q_{n+1} have no integer content. We will do this by way of an example.

Example 4.1. Consider $I = [-1/2, 1/2]$. Let $p_1(x) = P(x) = x^2$, $q_1(x) = Q(x) = 1 - 4x^2$, and $e = f = 1$. We see then that p_1, q_1 and P, Q are both Gorshkov-Wirsing pairs. Consider $P(U_n(x))$ for a generic p and q . We see that $P(U_n(x)) = \frac{1}{4} \times \frac{(p_n - q_n)^2}{(p_n + q_n)^2}$. This gives us that $p_{n+1} = (p_n - q_n)^2$.

Next consider $Q(U_n(x))$ for a generic p_n and q_n . We get $Q(U_n(x)) = 1 - \left(\frac{p-q}{p+q}\right)^2 = \frac{4p_n q_n}{(p_n + q_n)^2}$. This gives us that $q_{n+1} = p_n q_n$. So, we see that we can pre-remove integer content in most cases. In the second case, there will be no other integer content that may creep into the calculation by accident. If there is integer content that should be removed, but is not, then we still have a valid Gorshkov-Wirsing sequence, although not normally as good.

Now, given such a sequence, it is important to consider how one computes $\lim a_n^{-1/\deg p_n}$. To simplify the discussion, we will assume that $e = f = 1$, although the proof is equally valid for arbitrary positive integers e and f . Let p_1, q_1 and P, Q be Gorshkov-Wirsing pairs. Let $P(x, y)$ and $Q(x, y)$ be the homogenization of P and Q , and C_p and C_q be constants such that $P(a \cdot x + b \cdot y, x + y)c_p$ and $Q(a \cdot x + b \cdot y, x + y)c_q$ have no integer content. I.e. for the example above we have $P(x, y) = (x - y)^2$ and $Q(x, y) = x \cdot y$. Then we see that $p_{n+1} = P(a \cdot p_n + b \cdot q_n, p_n + q_n)c_p$, and similarly for q_{n+1} . This method for computing p_{n+1} and q_{n+1} allows us to compute the lead coefficient of p_{n+1} and q_{n+1} . Namely $a_{n+1} = P(a \cdot a_n + b \cdot b_n, a_n + b_n)c_p$ and $b_{n+1} = Q(a \cdot a_n + b \cdot b_n, a_n + b_n)c_q$.

Theorem 4.2. *With a_n and b_n defined as above, and $d_n = \deg(p_n) = \deg(q_n)$. Then $d_n = \deg(p_1) \deg(P)^{n-1}$. Further $\lim |a_n|^{-1/d_n}$ and $\lim |b_n|^{-1/d_n}$ are both well defined, and equal.*

Proof. Let p_1 and q_1 be degree k , and P and Q be degree m . We easily see by induction that $\deg p_n = \deg q_n = \deg p_1 (\deg P)^{n-1} = k \cdot m^{n-1}$.

For now assume that $|a_n| \leq |b_n|$. Then we get

$$\begin{aligned} a_{n+1} &= P(a \cdot a_n + b \cdot b_n, b_n + a_n) c_P \\ &= P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P b_n^m \\ b_{n+1} &= Q(a \cdot a_n + b \cdot b_n, b_n + a_n) c_Q \\ &= Q\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_Q b_n^m \end{aligned}$$

We see that $P(ax + b, 1 + x)$ and $Q(ax + b, 1 + x)$ are bounded for $x \in [-1, 1]$, say by M . Then $a_{n+1} \leq M b_n^m$ and $b_{n+1} \leq M b_n^m$. We see that degree of p_n and q_n is $d_n = k \cdot m^{n-1}$. Now, $|a_{n+1}|^{-1/d_{n+1}} \leq \prod M^{-1/(k \cdot m^i)} |b_0|$, and similarly for b_{n+1} . This is easily seen to converge by taking logarithms of the right hand side. As the $|a_n|^{-1/d_n}$ are increasing, and bounded, they converge.

Now, to see they are equal, we noticed that

$$\begin{aligned} a_{n+1}^{-1/d_{n+1}} &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P b_n^m \right)^{-1/d_{n+1}} \\ &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P \right)^{-1/d_{n+1}} b_n^{-m/d_{n+1}} \\ &= \left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P \right)^{-1/d_{n+1}} b_n^{-1/d_n} \end{aligned}$$

Here $\left(P\left(a \cdot \frac{a_n}{b_n} + b, 1 + \frac{a_n}{b_n}\right) c_P \right)^{-1/d_{n+1}}$ goes to 1 as n goes to infinity, from which the result follows.

We notice a similar argument holds if $|a_n| > |b_n|$ for some or all of the n . \square

5. SEARCHING FOR GENERALIZED GORSHKOV-WIRSING POLYNOMIALS

The next two lemmas combine to give us an algorithm to find $U(x) \in \mathcal{U}[a, b]$. This is a heuristic technique only, and is not proven to get all good values in $\mathcal{U}[a, b]$. An additional benefit of this technique is that it find many ‘‘good’’ polynomials for an interval, in the sense that $a_k^{-1/k}$ is small, and all of its roots are in the interval in question.

Lemma 5.1. *If $p(x)$ has all of its roots in $[a, b]$ and $U(x) \in \mathcal{U}[a, b]$ then $\text{numer}(p(U(x)))$ has all of its roots in $[a, b]$.*

Lemma 5.2. *If $U(x) \in \mathcal{U}[a, b]$ then $\text{numer}(U(x) - x)$ has all of its roots in $[a, b]$.*

To find $U(x) \in \mathcal{U}[a, b]$ we first let \mathcal{Q} be a set of irreducible polynomials with all of their roots in the interval $[a, b]$. Included in this set are the two linear polynomials with roots at a and b . One easy way to derive a starting set of polynomials is by

using LLL. From this set, we find the set of all pairs (p, q) that satisfy the conditions of Theorem 4.1, and call this set \mathcal{P} .

We use the set \mathcal{P} and \mathcal{Q} and Lemma 5.1 to find new polynomials with all of their roots in $[a, b]$ and add these to the set \mathcal{Q} . We use the set \mathcal{P} and Lemma 5.2 to find new polynomials with all of their roots in $[a, b]$ and add these to the set \mathcal{Q} . Normally, we limit the set \mathcal{Q} in some way, based on Lemma 1.1, and on the degree of the polynomials. Using this new set \mathcal{Q} we derive a new set \mathcal{P} . We repeat this procedure until we no longer find any new polynomials or pairs given some set of restriction.

Example 5.1. Consider the interval $[-1/2, 1/2]$ and the set:

$$\mathcal{Q} = \{2x - 1, x, 2x + 1\}.$$

From this we derive the pairs:

$$\mathcal{P} = \{[x^2, (1 - 2x)(2x + 1)]\}.$$

From Lemma 5.1 this gives us the new polynomial:

$$\{5x^2 - 1\}.$$

From Lemma 5.2 we get the new polynomials:

$$\{5x^2 - 1, 3x - 1\}.$$

So our new set \mathcal{Q} becomes:

$$\mathcal{Q} = \{x, 2x - 1, 2x + 1, 5x^2 - 1, 3x - 1\}.$$

From this we derive the pairs:

$$\begin{aligned} \mathcal{P} = & \{[x^2, (1 - 2x)(2x + 1)], \\ & [(5x^2 - 1)^2, (1 - 2x)(2x + 1)(3x - 1)^2], \\ & [(5x^2 - 1)^2, (1 - 2x)(2x + 1)x^2], \\ & [(3x - 1)^2, (1 - 2x)(2x + 1)], \\ & [(1 - 2x)x^2, (2x + 1)(3x - 1)^2]\}. \end{aligned}$$

From here, we can keep repeating the process, eliminating pairs and polynomials that don't meet some sort of criteria, and eventually deciding that we are done.

6. SOME UPPER AND LOWER BOUNDS FOR INTERVALS

Using the techniques of Section 5 we compute upper and lower bounds for a number of sets. The precise pairs, as well as more complete tables (up to denominator 5) can be found at [14]. The purpose of these tables is to compare Gorshkov-Wirsing pairs with other methods. The Gorshkov-Wirsing pairs were found up to degree $n = 11$. When possible, this was extended beyond $n = 11$.

We looked at five different types of intervals. Very little will be said about Tables 2, 4, 6, or 8. These were included for reference only.

We see that Tables 3, 5, and 9, that the Gorshkov-Wirsing pairs give a tighter lower bound than the other methods. This is not surprising given that these other methods are designed for Farey intervals. We see in Table 7 that the Gorshkov-Wirsing pairs give the same values as that of Flammang, Rhin and Smyth, and that of Flammang. In these cases, this is just alternate ways of computing the same limit point.

Tables 10 and 11 demonstrate what happens for intervals close to $[0, 1]$, but where the actual interval has very high denominator. We see that LLL, Simplex, and the method of Habsieger and Salvy do not suffer any ill effects from this. The Gorshkov-Wirsing method does suffer some effects, but not as severely as Amoroso. The methods of Flammang, and those of Flammang, Rhin and Smyth were not included, as they are only relevant for Farey intervals.

7. CONCLUSIONS

Borwein and Erdelyi [5] showed that $[0, x]$ is continuous and constant for x near 1. Further they showed that there exists a δ such that for all $0 \leq a < \delta$ that $t_{\mathbb{Z}}[-a, 1+a] = t_{\mathbb{Z}}[0, 1]$. Unfortunately their method does not easily allow an explicit computation of δ . It would be interesting to use these Gorshkov-Wirsing pairs for $a \in \mathbb{Q}$ with $a < \delta$, to see if better lower bounds could be found for $t_{\mathbb{Z}}[0, 1]$.

In this same paper, Borwein and Erdelyi also showed that the limit coming from these Gorshkov-Wirsing pairs cannot be tight for the interval $[0, 1]$. This argument, in fact, holds when the end points of the interval are roots of critical polynomials. This is not always the case for some of the problems looked at in this paper, and it is possible (although the author does not consider it likely) that the limit coming from these Gorshkov-Wirsing pairs is tight in these cases.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA,
N2L 3G1

E-mail address: kghare@math.uwaterloo.ca

Degree n	Polynomial $p(x)$	$t_{z_n}[0, 1]$
2	$x(1-x)$	1/2
3	$x(1-x)(2x-1)$	1/2.18...
4	$x^2(1-x)^2$ or $x(1-x)(2x-1)^2$	1/2
5	$x^2(1-x)^2(2x-1)$	1/2.23...

TABLE 1. Small degree Integer Chebyshev polynomials

Interval	LLL	SIMPLEX	HS	Amoroso	Lower	# CP
[-1, 1]	1/1.5314	1/1.5334	1/1.4772	1/1.4520	1/1.5417	8
[-1/2, 1/2]	1/2.3559	1/2.3619	1/2.1822	1/1.4520	1/2.3768	9
[-1/3, 1/3]	1/3.2522	1/3.2617	1/3.0000	1/1.3887	1/3.2842	7
[-2/3, 2/3]	1/1.8820	1/1.8883	1/1.7237	1/1.3887	1/1.9845	5
[-1/4, 1/4]	1/4.1921	1/4.2025	1/4.0000	1/1.1097	1/4.2260	6
[-3/4, 3/4]	1/1.7897	1/1.7935	1/1.7237	1/1.1097	1/1.9653	3

TABLE 2. Upper Bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \text{int}(I)$, I symmetric

Interval	Upper	GW	FRS	Flammang	Amoroso
[-1, 1]	1/1.5334	1/1.5417	1/2.3768	1/2.3768	1/1.7024
[-1/2, 1/2]	1/2.3619	1/2.3768	1/3.5132	1/3.5132	1/3.4048
[-1/3, 1/3]	1/3.2617	1/3.2842	1/4.5940	1/4.5940	1/4.7345
[-2/3, 2/3]	1/1.8883	1/1.9845	1/3.5132	1/3.5132	1/3.1860
[-1/4, 1/4]	1/4.2025	1/4.2260	1/5.6494	1/5.6494	1/5.7853
[-3/4, 3/4]	1/1.7935	1/1.9653	1/3.5132	1/3.5132	1/3.0000

TABLE 3. Lower Bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \text{int}(I)$, I symmetric

Interval	LLL	SIMPLEX	HS	Amoroso	Lower	# CP
[-1/2, 1]	1/1.8133	1/1.8190	1/1.6055	1/1.4520	1/1.8743	3
[-1/3, 1]	1/2.0248	1/2.0309	1/1.8899	1/1.3887	1/2.0617	8
[-2/3, 1]	1/1.6560	1/1.6657	1/1.4142	1/1.3887	1/1.7410	3
[-1/3, 1/2]	1/2.6978	1/2.7094	1/2.1743	1/1.3887	1/2.7788	4
[-1/2, 2/3]	1/2.0303	1/2.0443	1/1.7411	1/1.3887	1/2.1865	4
[-1/3, 2/3]	1/2.2740	1/2.2801	1/1.9332	1/1.3887	1/2.4537	2

TABLE 4. Upper Bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \text{int}(I)$, I non-symmetric

Interval	Upper	GW	FRS	Flammang	Amoroso
[-1/2, 1]	1/1.8190	1/1.8743	1/2.3768	1/2.3768	1/2.2151
[-1/3, 1]	1/2.0309	1/2.0617	1/2.3768	1/2.3768	1/2.3187
[-2/3, 1]	1/1.6657	1/1.7410	1/2.3768	1/2.3768	1/2.1307
[-1/3, 1/2]	1/2.7094	1/2.7788	1/3.5132	1/3.5132	1/3.6580
[-1/2, 2/3]	1/2.0443	1/2.1865	1/3.5132	1/3.5132	1/3.1860
[-1/3, 2/3]	1/2.2801	1/2.4537	1/3.5132	1/3.5132	1/3.1860

TABLE 5. Lower Bounds for $t_{\mathbb{Z}}(I)$, for $0 \in \text{int}(I)$, I non-symmetric

Interval	LLL	SIMPLEX	HS	Amoroso	Lower	# CP
[0, 1]	1/2.3546	1/2.3613	1/2.2361	1/2.3264	1/2.3768	10
[0, 1/2]	1/3.4689	1/3.4813	1/3.1923	1/2.3264	1/3.5132	8
[1/3, 1/2]	1/5.8364	1/5.8614	1/5.3126	1/2.3264	1/5.9112	7
[0, 1/3]	1/4.5235	1/4.5444	1/4.1930	1/2.3264	1/4.5940	7

TABLE 6. Upper Bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \text{int}(I)$, I Farey

Interval	Upper	GW	FRS	Flammang	Amoroso	BE
[0, 1]	1/2.3613	1/2.3768	1/2.3768	1/2.3768	1/2.4142	1/2.8750
[0, 1/2]	1/3.4813	1/3.5132	1/3.5132	1/3.5132	1/3.6580	1/3.9167
[1/3, 1/2]	1/5.8614	1/5.9112	1/5.9112	1/5.9112	1/8.3648	
[0, 1/3]	1/4.5444	1/4.5940	1/4.5940	1/4.5940	1/4.7345	1/4.9375

TABLE 7. Lower Bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \text{int}(I)$, I Farey

Interval	LLL	SIMPLEX	HS	Amoroso	Lower	# CP
[0, 2/3]	1/2.7940	1/2.8056	1/2.3811	1/2.3264	1/2.8804	5
[1/3, 2/3]	1/3.8577	1/3.8707	1/3.4641	1/2.3264	1/3.8920	6

TABLE 8. Upper Bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \text{int}(I)$, I non-Farey

Interval	Upper	GW	FRS	Flammang	Amoroso
[0, 2/3]	1/2.8056	1/2.8804	1/3.5132	1/3.5132	1/3.1860
[1/3, 2/3]	1/3.8707	1/3.8920	1/5.9112	1/5.9112	1/7.2426

TABLE 9. Lower Bounds for $t_{\mathbb{Z}}(I)$, for $0 \notin \text{int}(I)$, I non-Farey

Interval	LLL	SIMPLEX	HS	Amoroso	Lower	# CP
[1/5, 4/5]	1/2.6391	1/2.6426	1/2.2134	1/2.0166	1/2.8590	3
[1/10, 9/10]	1/2.3532	1/2.3623	1/2.1822	1/1.0630	1/2.7502	8
[1/20, 19/20]	1/2.3532	1/2.3622	1/2.1822	1/.66772	1/2.8949	8
[-1/5, 6/5]	1/2.0441	1/2.0486	1/2.0000	1/1.0257	1/2.2502	3
[-1/10, 11/10]	1/2.2965	1/2.2971	1/2.0000	1/.78152	1/2.5474	4
[-1/20, 21/20]	1/2.3543	1/2.3619	1/2.1822	1/.57228	1/2.6246	10

TABLE 10. Upper Bounds for $t_{\mathbb{Z}}(I)$, for $I \subset [0, 1]$ and $I \supset [0, 1]$, Exact Computation

Interval	Upper	GW	Amoroso
[1/5, 4/5]	1/2.6426	1/2.8590	1/8.0189
[1/10, 9/10]	1/2.3623	1/2.7502	1/13.953
[1/20, 19/20]	1/2.3622	1/2.8949	1/24.906
[-1/5, 6/5]	1/2.0486	1/2.2502	1/7.1367
[-1/10, 11/10]	1/2.2971	1/2.5474	1/13.117
[-1/20, 21/20]	1/2.3619	1/2.6246	1/24.351

TABLE 11. Lower Bounds for $t_{\mathbb{Z}}(I)$, for $I \subset [0, 1]$ and $I \supset [0, 1]$, Exact Computation