MATH 249, WINTER 2017, ASSIGNMENT 6

This assignment is due Friday March 17 at 3pm. It should be submitted using crowdmark. The crowdmark instructions are the same as before.

Questions.

(1) **(6 points)** Read section 8.4 of the math 239 notes (on Hall's theorem).

(a) Find an example of a graph G and a subset S of vertices of G so that for all $D \subseteq S$, $|N(D)| \geq |D|$ but there is no matching of G saturating S. *This shows that the bipartite hypothesis is necessary in Hall's theorem.*

(b) What if we modify the definition of $N(D)$ to not include vertices which are within D itself, call this $N_1(D)$. Then show that the non-bipartite analogue of Hall's theorem using $N_1(D)$ fails in the other direction.

(2) **(4 points BONUS)** Let $\mathcal{H}$ be the Connes-Kreimer Hopf algebra of rooted trees and let $\mathcal{T}$ be the set of rooted trees (so as an algebra $\mathcal{H} = \mathbb{Q}[\mathcal{T}]$). Suppose we have two functions $f : \mathcal{H} \to \mathcal{H}$ and $g : \mathcal{T} \to \mathcal{H}$. We can define the *convolution* of f and g, written $f \star g : \mathcal{T} \to \mathcal{H}$ by

$$f \star g = m(f \otimes g)\Delta$$

where m takes the two sides of a tensor and returns the single element of $\mathcal{H}$ which is the product of the two sides (this is the multiplication of $\mathcal{H}$ viewed in this tensor language). That is, to calculate $f \star g$ on an element $t \in \mathcal{T}$, first apply the coproduct to t, then apply f to the left hand side of all of the resulting tensors and apply g to the right hand side, then forget the tensor signs so we are left with a sum of products of the form $f(a)g(b)$.

Prove that there exists a unique function $S : \mathcal{T} \to \mathcal{H}$ with the property that $\mathrm{id} \star S$ is the function which takes an element of $\mathbb{Q}[\mathcal{T}]$ to its constant term as a polynomial.

Hint: unwind the meaning of $\mathrm{id} \star S$ *being this function to get a recursive definition for* S*. This* S *is called the* antipode *of* $\mathcal{H}$*.*

(3) **(5 points)** In the graph Hopf algebra where the coproduct runs over bridgeless subgraphs, calculate

$$\Delta(K_4)$$

For this application please keep loops and multiple edges that appear during contraction.

(4) **(8 points)** Let n be a nonnegative integer. A *composition* of n is a sequence $(m_1, m_2, \ldots, m_k)$ of positive integers such that $m_1 + m_2 + \cdots + m_k = n$. The number of compositions of n is 2^{n-1}. Now you give a few different proofs of this fact.

(a) Give a proof that the number of compositions of n is 2^{n-1} using a bijection with binary strings. Show the details.

(b) Give a proof that the number of compositions of n is 2^{n-1} by building compositions of n out of smaller compositions. Show the details.

(c) Give a bijection between compositions and multisets.

(5) **(6 points)** Let $n \geq 0$ and $t \geq 2$ be integers. This question concerns the identity

$$\binom{n+t-1}{t-1} = \sum_{j=0}^{n} \binom{j+t-2}{t-2}$$

(a) Give a combinatorial proof of this identity.
(b) Give an algebraic proof of this identity.

(6) Recall that a matroid is a pair $M = (E, \mathcal{I})$, where E is a finite 'ground' set and $\mathcal{I}$ is a collection of subsets of E such that

- $\varnothing \in \mathcal{I}$,
- if $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$, and
- for all $I, J \in \mathcal{I}$ with $|I| < |J|$ then there exists $e \in J - I$ such that $I \cup \{e\} \in \mathcal{I}$.

(a) **(2 points)** Given $e \in E(M)$, let $M \backslash e = (E, \{I \in \mathcal{I} \colon e \notin I\})$ and $M/e = (E, \{I \subseteq E - \{e\} \colon I \cup \{e\} \in \mathcal{I})$. Show that $M \backslash e$ and M/e are matroids.
(b) **(3 points)** Let $\mathcal{I}^*$ be the collection of subsets of E whose complement contains a maximal element of $\mathcal{I}$. (In other words, the maximal sets in $\mathcal{I}^*$ are the complements of the maximal sets in $\mathcal{I}$.) Show that M^* is a matroid, and that $(M/e)^* = M^* \backslash e$ for all $e \in E$.
(c) **(3 points BONUS)** For a real matrix A with columns indexed by E, let $M(A)$ denote the matroid $(E, \mathcal{I})$, where $\mathcal{I}$ is the collection of subsets of E for which the corresponding set of columns of A is linearly independent. Show that if B is an $r \times (n-r)$ matrix and $A = [I_r | B]$, then $M(A)^* = M(A^*)$, where A^* denotes the matrix $[B^T | I_{n-r}]$ whose columns are also indexed by E. *Note that you've seen this before on an assignment. What is the connection?*