MATH 249, WINTER 2017, ASSIGNMENT 5

This assignment is due Friday February 17 at 3pm. It should be submitted using crowdmark. The crowdmark instructions are the same as before.

Questions.

(1) **(3 points BONUS)** Draw the Petersen graph on a torus (with no edge crossings).

(2) **(5 points)** Let G be a planar graph with n vertices and m edges.
  (a) Let T be a spanning tree of G. Prove that the edges of G^* which do not correspond to edges of T form a spanning tree of G^*.
  (b) Let $[I_{n-1}|A]$ be an $n-1 \times m$ matrix. Suppose that the submatrix formed by the columns indexed by $J \subset \{1, 2, \ldots, m\}$ with $|J| = n-1$ has nonzero determinant. Prove that the columns not indexed by J in $[-A^T|I_{m-n+1}]$ also have nonzero determinant.

  In fact you just proved the same thing twice because if we take the signed incidence matrix of G and remove a row and choose the order on the edges so that the first $n-1$ columns form a spanning tree then we can row reduce this matrix to the form $[I_{n-1}|A]$. In this case the signed incidence matrix of G^* with one row removed is row-equivalent to $[-A^T|I_{m-n+1}]$.

(3) **(5 points)** The *girth* of a graph is the length of a shortest cycle.
  (a) Prove that every planar graph with girth at least 4 is 4-colourable (don't use the 4-colour theorem).
  (b) What girth is required to guarantee that a planar graph with that girth is 3-colourable using analogous techniques to the previous part and what we did in class?

(4) **(6 points)**
  (a) Let G be a connected planar graph which is isomorphic to its dual. Prove that $|E(G)| = 2|V(G)| - 2$.
  (b) Find a connected planar graph on 6 vertices which is isomorphic to its dual.
  (c) How many non-isomorphic connected planar multigraphs on 4 vertices which are isomorphic to their duals are there?

(5) **(4 points)**
  (a) Show that if two opposite corner squares of a chessboard (chessboards are 8×8) are removed then the resulting board cannot be covered with 31 dominoes each of which covers two adjacent squares.
  (b) What did the previous part have to do with things we discussed in class?

(6) **(5 points)**
  (a) Let G be a bipartite graph, and let Δ be the maximum degree of any vertex in G. Show that G has a matching with at least $|E(G)|/\Delta$ edges.
  (b) Give an example of a non-bipartite graph for which this lower bound on the size of a matching is not true.