MATH 249, WINTER 2017, ASSIGNMENT 4

This assignment is due Friday February 10 at 3pm. It should be submitted using crowdmark. The crowdmark instructions are the same as before.

Questions.

(1) **(5 points)** Suppose G is a k-regular graph on v vertices with adjacency matrix A. Let τ be the least eigenvalue of A, let I be the identity matrix and J the all ones matrix. Show that the matrix

$$A - \tau I - \frac{k-\tau}{v} J$$

has all nonnegative eigenvalues. *Hint: use the fact that for a symmetric matrix we can choose orthogonal eigenvectors and consider the other eigenvalues of A.*

(2) **(3 points)** Let Φ be the Kirchhoff polynomial and let

$$\Psi = \sum_{T} \prod_{e \notin T} a_e$$

where the sum runs over spanning trees of G and the conventions are otherwise the same as for the Kirchhoff polynomial. Prove that

$$\left(\prod_{e \in E(G)} a_e \right) \Phi \left(\frac{1}{e_1}, \dots, \frac{1}{e_{|E(G)|}} \right) = \Psi(e_1, e_2, \dots, e_{|E(G)|})$$

(3) **(10 points)** Let G be a graph with oriented incidence matrix E. Let Λ be the diagonal matrix with (i,i)th entry the edge variable a_i. Let $\widetilde{E}$ be E with one row removed. Consider the block matrix

$$M = \begin{bmatrix} \Lambda & \widetilde{E}^T \\ -\widetilde{E} & 0 \end{bmatrix}$$

(a) Prove that $\det(M) = \Psi$ (where Ψ is as defined in the previous question.)
(b) Interpret $\det(M)|_{a_1=0}$ in two ways, first as Ψ for some graph and second in terms of spanning forests of G with edge 1 removed.
(c) Let $M_{1,2}$ be the matrix M with the first row and second column removed. Suppose edges 1 and 2 meet at a vertex v. Characterize which monomials appear with nonzero coefficients in $\det(M_{1,2})$ using spanning forests of G with edges 1 and 2 removed.

(4) **(4 points)** Give two different proofs that the Petersen graph is nonplanar. At least one of them cannot use results that we didn't prove (unless you prove them yourself).

(5) (a) **(5 points)** Which circulants (with any number of gap parameters) on 8 vertices are planar?
(b) **(3 points BONUS)** What can you say abound the planarity of a circulant graph given only the number of gap parameters? (You can assume the gap

parameters are distinct and at most $n/2$.) If you are feeling ambitious you could try to fully characterize the planar circulants.

(6) **(4 points)** Give an example of a planar embedding of a graph graph in which every vertex has degree 3 or 4, every face has degree 4 or 5, but which the graph contains a 3-cycle.

(7) **(4 points)** Show that a regular connected graph with a planar embedding where each face is a 4-cycle has at most eight vertices.