COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 9 SUMMARY

WINTER 2018

Summary

The first thing we did today was to compare our hand computations of numbers of small polygon gluings by genus to the table in Harer and Zagier (exact reference is given below). Then we worked out using Wick's theorem that

$$\int_{\mathrm{Herm}(N)} \mathrm{tr}(N^4)d\mu(H) = 2N^3 + N$$

which corresponds to the fact that there are two gluings of the square of genus 0 and one of genus 1. The general result is

Proposition 1.

$$\int_{Herm(N)} tr(N^{2m})d\mu(H) = \sum_{g\geq 0} \epsilon_g(m)N^{m+1-2g}$$

where $\epsilon_g(m)$ is the number of polygon gluings of a $2m$-gon which result in a genus g surface.

The proof is to apply Wick's theorem to $\langle h_{i_1,i_2}h_{i_2,i_3}\cdots h_{i_{2m},i_1}\rangle$. Each term in the Wick expansion forces equalities of the i_j (since most of the $\langle h_{i,j}h_{k,l}\rangle$ are 0). If we label the vertices of the $2m$-gon with $i_1, i_2, \ldots$ then these equalities are identical to those resulting from identifying the vertices according to this gluing. For the term in the Wick expansion the number of free variables from this system of equations is the power of N contributed by that term, while for the polygon the number of free variables is the number of vertices in the embedded graph. Euler'f formula then completes the proof.

The last thing we did today was to define combinatorial maps. We already had the idea that a combinatorial map is a graph with a cyclic order around each vertex and that this captures the information of an embedded graph. The usual way to implement this idea is perhaps not what one would first think of, but it's quite useful and worth knowing.

Definition 2. *A* combinatorial map *is two permutations of the same underlying set,* (σ, α) *where α is a fixed-point free involution.*

The *underlying graph* of a combinatorial map $M = (\sigma, \alpha)$ is the graph with vertices given by the orbits of σ, edges given by the orbits of α and a vertex and edge incident if they share an element.

The idea is that the underlying set being permuted is the set of half-edges of the graph. α then pairs the half-edges into edges, and the cycles of σ give the vertices. The underlying graph forgets the cyclic order of these cycles, just keeping which elements they involve.

Next time

Next class we will say more about combinatorial maps and then return to the polygon gluings (1-face maps) and begin the derivation of a closed form for the generating function.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)

Harer and Zagier *The Euler characteristic of the moduli space of curves* Invent. math. **85**, 457-485 (1986)