COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 8 SUMMARY

WINTER 2018

Summary

Today we began working on a very nice combinatorial use of Wick's theorem, namely counting combinatorial maps using matrix integrals.

The first step is to set up the kind of matrix integrals we'll need. Fix N. We are going to be working with a Gaussian measure on $\mathbb{R}^{N^2}$ and it will be convenient for both large and small reasons to use matrices to hold the variables. However we won't do this in the most naive way, rather we'll use Hermitian matrices. Let $H = (h_{i,j})$ be an $N \times N$ Hermitian matrix (that is $H = \overline{H}^t$) and let Herm(N) be the space of all $N \times N$ Hermitian matrices. This space has real dimension N^2 and we'll write $h_{i,i} = x_{i,i}$, $h_{i,j} = x_{i,j} + iy_{i,j}$ for $i < j$ with the xs and ys the N^2 real variables.

Here is a collection of definitions and easy computations that we'll need:

$$d\nu(H) = \prod_{i=1}^{N} dx_{i,i} \prod_{i<j} dx_{i,j} dy_{i,j}$$

is the volume form we'll use.

$$\operatorname{tr}(H^2) = \sum_{i=1}^{N} x_{i,i}^2 + 2\sum_{i<j}(x_{i,j}^2 + y_{i,j}^2)$$

which has matrix B which is diagonal with N 1s and then $N^2 - N$ 2s on he diagonal. This will be the matrix defining our Gaussian measure and so

$$d\mu(H) = \frac{e^{-\frac{1}{2}\operatorname{tr}(H^2)} d\nu(H)}{\sqrt{2}^N \sqrt{\pi}^{N^2}}$$

With this measure we calculate

$$\begin{aligned}
\langle x_{i,i}^2 \rangle &= 1 \\
\langle x_{i,j}^2 \rangle = \langle y_{i,j}^2 \rangle &= \frac{1}{2} \quad \text{for } i < j \\
\langle h_{i,j} h_{j,i} \rangle &= 1 \\
\langle h_{i,j} h_{k,l} \rangle &= 0 \quad \text{for } (i,j) \neq (k,l)
\end{aligned}$$

That will be enough matrix integral set up for the moment. Next we need to define and understand the combinatorial objects we will be counting. These will be *polygon gluings*. Given a chord diagram on $2n$ vertices we can reinterpret this by taking the vertices of the diagram as the edges of a $2n$-gon and then when two of these are joined by a chord, glue them (in the orientable way) to make a surface out of the $2n$-gon. The boundary of the $2n$-gon becomes a graph embedded on the surface. The embedded graph as exactly one face (namely the $2n$-gon itself) and so all faces are disks; therefore the genus of the surface is the same as the genus of the graph. We can calculate the genus of the graph by Euler's formula:

$$2 - 2g = V - E + F$$

In this case $F = 1$ and $E = n$ (because each edge of the polygon is glued to one other edge forming an edge of the graph). So we end up with

$$g = \frac{n - 1 + V}{2} + 1$$

It's worth seeing some examples so I asked you to try all the rooted chord diagrams on 4 and 6 vertices and build the resulting surfaces (and determine the genus in each case). We'll get back to that computation next time.

Next time

Next class we will look at counting polygon gluings by n and g. First we'll work on our intuition by looking at a table which extends your computation and second we'll see how to use Hermitian Gaussian integrals to do it.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)