COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 6 SUMMARY

WINTER 2018

Summary

Today we finally proved Wick's theorem.

First we had a lemma (from last time but I didn't put in the summary last time). Interpret it all with multi-indices.

Lemma 1. *Suppose* $\log\left(1 + \sum_{|\alpha|>0} m_\alpha \frac{t^\alpha}{\alpha!}\right) = \sum_{|\alpha|>0} s_\alpha \frac{t^\alpha}{\alpha!}$ *as formal power series. Then*

$$m_\alpha = \sum_{k=1}^{|\alpha|} \frac{1}{k!} \sum_{\substack{\beta^{(1)}+\beta^{(2)}+\cdots+\beta^{(k)}=\alpha \\ |\beta^{(i)}|>0}} \frac{\alpha!}{\beta^{(1)}!\cdots\beta^{(k)}!} s_{\beta^{(1)}} s_{\beta^{(2)}} \cdots s_{\beta^{(k)}}$$

The proof is simply to exponentiate the defining equation. We will use this equation where the m_α are moments and the s_α are semi-invariants. This is particularly nice for the Gaussian measures where the only nonzero semi-invariants are the quadratic ones. Specifically for a Gaussian measure with covariance matrix C:

$$\log(\phi(t)) = -\frac{1}{2} t^t C t$$

Where ϕ is the Fourier transform (so the exponential generating function for the moments but with the i multiplying the indeterminate). In the case that C is singular we can use this equation to define the Gaussian measure and so we have Gaussian measures defined for any C.

Then Wick's theorem (or Isserlis' theorem) is

Theorem 2. *Let $f_1, f_2, \ldots, f_{2n}$ be any linear functions (not necessarily distinct) of $x_1, x_2, \ldots, x_k$. Then for any Gaussian measure on $\mathbb{R}^k$*

$$\langle f_1 f_2 \cdots f_{2n} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_n} f_{q_n} \rangle$$

where the sum is over all pairings, specifically, over all permutations $p_1 q_1 p_2 q_2 \cdots p_n q_n$ of $\{1, 2, \ldots, 2n\}$ such that $p_1 < p_2 < \cdots p_n$ and $p_i < q_i$ for $1 \leq i \leq n$.

A sketch of the proof is as follows (we went through the details in class): The f_i define a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^{2n}$. Let A be the matrix of this transformation and let C be the covariance matrix of the Gaussian measure on $\mathbb{R}^k$ given in the question. Then the linear transformation given by the f_i induces a Gaussian measure on $\mathbb{R}^{2n}$ with covariance matrix ACA^t. Now use the lemma to compute the moment $\langle f_1(x) f_2(x) \cdots f_{2n}(x) \rangle$. This corresponds to multi-index $(1, 1, 1, \ldots, 1)$ but since the measure is Gaussian, the only nonzero semi-inavariants that can contribute on the right hand side of the lemma are those with multi-index of the form $(0, \ldots, 0, 1, 0 \ldots, 0, 1, 0 \ldots, 0)$.

Note a few things. First, any $\langle f_1 f_2 \cdots f_{2n-1} \rangle = 0$ by the same argument. Second, this proof is combinatorial and makes sense at the level of formal power series. Third, this is typically where graphs, chord diagrams and pairings come from in QFT.

We finished the class with some modifications of our running example.

Next time

Next class we will start working on counting combinatorial maps by matrix integrals. This will give us a use of Wick's theorem that actually uses the multivariateness of it.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)