COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 6 SUMMARY

WINTER 2018

Summary

Today we looked at Gaussian measures in $\mathbb{R}^k$ and defined some probability vocabulary that we'll need.

Given a real symmetric $k \times k$ matrix B (for now also assume that $\det(B) \neq 0$) we define a Gaussian measure on $\mathbb{R}^k$ from B via

$$\frac{\det B}{\sqrt{2\pi}^k} \exp(-\frac{1}{2}x^t Bx)dx_1 dx_2 \cdots dx_k$$

(Note a typo corrected from lecture). We proved this was normalized by using the fact that real symmetric matrices can be diagonalized by an orthogonal matrix. Then the integral over $\mathbb{R}^k$ breaks into k integrals over $\mathbb{R}$ which are all of the form we already understand.

The inverse of B, $C = (c_{i,j}) = B^{-1}$ is called the *covariance matrix*. If $B = C = I$ then the associated Gaussian measure is standard. We'll use angled brackes for the mean, that is for any measure μ on X and $f : X \to \mathbb{R}$, $\langle f \rangle = \int_X f(x)d\mu(x)$. Then we know some facts:

- For the Gaussian measure on $\mathbb{R}$:

$$\langle 1 \rangle = 1$$
$$\langle x \rangle = 0$$
$$\langle x^k \rangle = \begin{cases} (k-1)!! & k \text{ even and } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

- For the Gaussian measure on $\mathbb{R}^k$:

$$\langle 1 \rangle = 1$$
$$\langle x_i \rangle = 0$$
$$\langle x_i x_j \rangle = c_{i,j}$$

  (try the computations yourself, the key again being to diagonalize).

Next we considered the exponential generating function of the moments in the 1-dimensional case. If we multiply the variable by i this is the Fourier transform or the characteristic function of the measure.

With multi-indices we get the same story for $\mathbb{R}^k$. Let

$$\phi(t) = 1 + \sum_{|\alpha|>0} m_\alpha \frac{t^\alpha}{\alpha!}$$

then write

$$\log(\phi(t)) = \sum_{|\alpha|>0} s_\alpha \frac{t^\alpha}{\alpha!}$$

The s_α are called the semi-invariants of the measure. There are two reasons that we will like them: first they make sense formally (as coefficients of the formal power series given above) and second they are particularly nice for Gaussian measures. That we will return to next time.

Next time

Next class we will finally give a formal power series proof of Wick's theorem.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)