# COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 5 SUMMARY

WINTER 2018

## Summary

We continued thinking about labelled and unlabelled combinatorial classes and constructions. We looked over the standard operations in both cases: (from Flajolet and Sedgewick)

In the unlabelled case

**Theorem I.1** (Basic admissibility, unlabelled universe). *The constructions of union, cartesian product, sequence, powerset, multiset, and cycle are all admissible. The associated operators are as follows.*

*Sum:* $\mathcal{A} = \mathcal{B} + \mathcal{C} \implies A(z) = B(z) + C(z)$

*Cartesian product:* $\mathcal{A} = \mathcal{B} \times \mathcal{C} \implies A(z) = B(z) \cdot C(z)$

*Sequence:* $\mathcal{A} = \textsc{Seq}(\mathcal{B}) \implies A(z) = \dfrac{1}{1 - B(z)}$

*Powerset:*
$$\mathcal{A} = \textsc{PSet}(\mathcal{B}) \implies A(z) = \begin{cases} \displaystyle\prod_{n \geq 1} (1 + z^n)^{B_n} \\ \exp\left( \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} B(z^k) \right) \end{cases}$$

*Multiset:*
$$\mathcal{A} = \textsc{MSet}(\mathcal{B}) \implies A(z) = \begin{cases} \displaystyle\prod_{n \geq 1} (1 - z^n)^{-B_n} \\ \exp\left( \displaystyle\sum_{k=1}^{\infty} \frac{1}{k} B(z^k) \right) \end{cases}$$

*Cycle:*
$$\mathcal{A} = \textsc{Cyc}(\mathcal{B}) \implies A(z) = \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log \frac{1}{1 - B(z^k)}.$$

*For the sequence, powerset, multiset, and cycle translations, it is assumed that $\mathcal{B}_0 = \emptyset$.*

The last column shows the operations as they translate to the ordinary generating functions. Note how intricate the generating function expressions are for the later operations.

Anything made of these operations along with $\mathcal{E}$ and $\mathcal{Z}$ (recursively or iteratively, or with a system of equations) we will say is *made of atoms*. Note that if an object is made of atoms then its size is the number of atoms. A *labelled* object is a pair of an object $c$ made of atoms and a function $f : \{\text{atoms of } c\} \to \{1, 2, \ldots, |c|\}$.

In the labelled case

**Theorem II.1** (Basic admissibility, labelled universe). *The constructions of combin torial sum, labelled product, sequence, set, and cycle are all admissible. Associat operators on EGFs are:*

| | | | |
|---|---|---|---|
| *Sum:* | $\mathcal{A} = \mathcal{B} + \mathcal{C}$ | $\Longrightarrow$ | $A(z) = B(z) + C(z),$ |
| *Product:* | $\mathcal{A} = \mathcal{B} \star \mathcal{C}$ | $\Longrightarrow$ | $A(z) = B(z) \cdot C(z),$ |
| *Sequence:* | $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$ | $\Longrightarrow$ | $A(z) = \dfrac{1}{1 - B(z)},$ |
| *— k components:* | $\mathcal{A} = \mathrm{SEQ}_k(\mathcal{B}) \equiv (\mathcal{B})^{\star k}$ | $\Longrightarrow$ | $A(z) = B(z)^k,$ |
| *Set:* | $\mathcal{A} = \mathrm{SET}(\mathcal{B})$ | $\Longrightarrow$ | $A(z) = \exp(B(z)),$ |
| *— k components:* | $\mathcal{A} = \mathrm{SET}_k(\mathcal{B})$ | $\Longrightarrow$ | $A(z) = \dfrac{1}{k!} B(z)^k,$ |
| *Cycle:* | $\mathcal{A} = \mathrm{CYC}(\mathcal{B})$ | $\Longrightarrow$ | $A(z) = \log \dfrac{1}{1 - B(z)},$ |
| *— k components:* | $\mathcal{A} = \mathrm{CYC}_k(\mathcal{B})$ | $\Longrightarrow$ | $A(z) = \dfrac{1}{k} B(z)^k.$ |

The last column shows the operations as they translate to exponential generating functions. They are much simpler.

The key to understanding labelled operations is $\star$. $\mathcal{B} \star \mathcal{C}$ is not just the set of ordered pairs of elements of $\mathcal{B}$ and $\mathcal{C}$, because the labellings would not be correct. Rather you take the set of all relabellings of the ordered pairs where the relabelling is compatible with the original labellings of the component (ie the order is preserved on each component). This does exactly the right thing for the exponential generating functions to multiply and so everything works out nicely.

Now we can make precise how counting weighted by $\frac{1}{|\mathrm{Aut}(c)|}$ is secretly labelled counting.

**Proposition 1.** *Let $\mathcal{D}$ be a labelled combinatorial class (with weights in $\mathcal{Z}_{\geq 0}$) and let $\alpha$ be the map which forgets the labelling. Let $\mathcal{C} = \alpha(\mathcal{D})$. Then*

$$\sum_{d \in \mathcal{D}} \frac{x^{|d|}}{|d|!} = \sum_{c \in \mathcal{C}} \frac{x^{|c|}}{|Aut(c)|}$$

*and*

$$\sum_{d\in\mathcal{D}} \frac{\alpha(d)x^{|d|}}{|d|!} = \sum_{c\in\mathcal{C}} \frac{cx^{|c|}}{|Aut(c)|}$$

The proof is elementary group theory: fix $c \in \mathcal{C}$, let $m = |\alpha^{-1}(c)|$. $\mathrm{Aut}(c)$ acts on the $|c|!$ labellings of the atoms of $c$. The orbits are the $m$ isomorphism classes of labellings, that is the elements of $\alpha^{-1}(c)$. So $|c|! = m|\mathrm{Aut}(c)|$. Now sum.

**Definition 2.** $\frac{1}{|Aut(c)|}$ *is called the* symmetry factor *of* $c$*.*

The point is that whenever you see counting weighted by symmetry factors then you can use the tools of labelled counting. In particular if $\mathcal{C}$ is a class of connected objects (eg conntected graphs or trees) then $\mathcal{D} = \mathsf{Set}(\mathcal{C})$ is the corresponding class of not necessarily connected objects (eg graphs, forests), and the exponential generating functions relate via $D(x) = \exp(C(x))$. We can apply this to our running example of the integral which gave a sum over graphs. Those graphs were potentially disconnected. If we only want the connected ones (still weighted by their symmetry factor) then just take the logarithm. This comes up a lot in QFT.

## Next time

Next class we will look at Gaussian measures in $\mathbb{R}^k$ and a formal power series proof of Wick's theorem.

## References

Flajolet and Sedgewick *Analytic Combinatorics* (Cambridge, 2009)