COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 4 SUMMARY

WINTER 2018

Summary

The first thing we did is expand out the first few terms of our running example and look at the coefficients we got. You can also see this done at the beginning of section 6.2 of `https://arxiv.org/pdf/1703.00840.pdf`. The answer is that the coefficients are $\frac{1}{|\mathrm{Aut}(G)|}$.

Whenever you see a sum over combinatorial objects where each is weighted by the inverse of the size of its automorphism group, you should immediately think that this is secret labelled counting. The goal of the next day or so (the rest of what we did today and much of what we will do on Friday) is to develop the theory so that this makes sense to everyone.

The first step in that direction is to define combinatorial classes and their generating functions. I will use curly letters for the combinatorial classes, eg $\mathcal{C}$, then subscript to mean just those elements of a given size, eg $\mathcal{C}_n = \{c \in \mathcal{C} : |c| = n\}$ (regardless of whether n is a multiindex or not). The one weird thing about the way I set things up is the notion of augmented generating functions:

Definition 1. *Let $\mathcal{C}$ be a combinatorial class. The* augmented generating function *of $\mathcal{C}$ is*

$$\sum_{c\in\mathcal{C}} c x^{|c|}$$

where we use multi-index notation if necessary, that is if $|c| = (|c|_1, |c_2|, \dots, |c_k|)$ then $x^{|c|} = x_1^{|c|_1} x_2^{|c|_2} \cdots x_k^{|c|_k}$, and for $k = 1$ we just have the usual power.

Now where does this object live? For now, instead of a ring for formal power series, lets not worry about multiplying coefficients and so just work in a vector space of formal power series. Then the augmented generating function lives in $\mathrm{Span}_F(\mathcal{C})[[x]]$ where F is the field we are over. $\mathrm{Span}_F(\mathcal{C})$ is finite formal sums of elements of $\mathcal{C}$. It is a graded vector space and so we can take its graded completion whenever it is useful.

Then you go from the augmented generating function to an ordinary generating function or an exponential generating function, or a perturbative expansion of a QFT amplitude, by an evaluation map. For example $\phi_{\mathrm{ordinary}}(c) = 1$ for all $c \in \mathcal{C}$, extended linearly will give the ordinary generating function, while $\phi_{\exp}(c) = \frac{1}{|c|!}$ will give the exponential generating function. The *Feynman rules* play this role for our QFT examples.

Finally we wrote down a few example combinatorial specifications. The last of them was a labelled specification so we haven't defined enough for it to make sense yet, though it should make sense by the end of Friday.

Next time

Next class we will continue looking at labelled and unlabelled counting so as to understand the symmetry factor, $\frac{1}{|\mathrm{Aut}(G)|}$.

References

To see our running example expanded out see `https://arxiv.org/pdf/1703.00840.pdf`. For the combinatorial classes and combinatorial specifications in a similar language to what I am using see Flajolet and Sedgewick *Analytic Combinatorics* (Cambridge, 2009)