COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 3 SUMMARY

WINTER 2018

Summary

Today we continued our discussion of the Gaussian measure on $\mathbb{R}^n$. We showed that its mean is 0, variance is 1 and the higher moments are 0 for odd moments and $(m-1)!!$ for even moments where $(2k-1)!! = (2k-1)(2k-3)\cdots(3)(1)$.

We had a digression into what $(2k-1)!!$ counts, the answer essentially being pairings (which you can think of as rooted chord diagrams on $2k$ vertices, or perfect matchings of K_{2k}, or fixed point free involutions of $\{1, 2, \ldots, 2k\}$ among others).

Returning to our running example we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq e^{(-\frac{1}{2}q^2+\frac{\lambda}{3!}q^3+Jq)} &= \sum_{\substack{i\geq 0\\ j\geq 0}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq e^{-\frac{1}{2}q^2} q^{j+3i}\right)\frac{J^j}{j!}\frac{\lambda^i}{(3!)^i i!} \\ &= \sum_{\substack{i\geq 0\\ j\geq 0}} \frac{J^j}{j!}\frac{\lambda^i}{(3!)^i i!}\begin{cases}(j+3i-1)!! & \text{if } j+3i \text{ is even}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

How do we interpret this combinatorially? We need to have $j+3i$ things to match. Each j comes with a power of J and each $3i$ comes with a power of λ, so for given values of j and i, draw j 1-valent vertices (each with a little end hanging off where edge will go) and i 3-valent vertices (each with 3 little ends hanging off), and we are counting matchings of these little ends, that is we are counting graphs with j 1-valent vertices and i 3-valent vertices.

There are some details: so far *everything* is labelled, the 1-valent vertices, the 3-valent vertices, and also the three half-edges out of each 3-valent vertex (you could equivalently label the edges with the caveat that every loop would need two possibilities). Some of these details we don't need to worry about in view of the factorials in the expansion. We'll get back to that next time.

Next time

Next class we will take a look by hand at the first few terms and say more about symmetries both in general and in this situation.

References

The Gaussian stuff can be found at the beginning of section 3.2 of "Graphs on surfaces and their applications" by Lando and Zvonkin (Springer 2004).