COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 34 SUMMARY

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Summary

Today we discussed denominator reduction.

This is an algorithm due to Francis Brown. In fact it is two related algorithms, one involving only polynomial manipulations and one an integration technique.

We can start integrating the Feynman period one integral at a time. The first two are calculus exercises.

$$\begin{aligned}\int_0^\infty \frac{da_1}{\Psi^2} &= \int_0^\infty \frac{da_1}{(\Psi_{G\backslash e_1}a_1 + \Psi_{G/e_1})^2} \\ &= -\frac{1}{(\Psi_{G\backslash e_1}a_1 + \Psi_{G/e_1})\Psi_{G\backslash e_1}}\Bigg|_{a_1=0}^{\infty} \\ &= \frac{1}{\Psi_{G\backslash e_1}\Psi_{G/e_1}}.\end{aligned}$$

To keep from going too crazy use the notation $\Psi^I = \Psi_{G\backslash I}$ and $\Psi_I = \Psi_{G/I}$. Then

$$\begin{aligned}\int_0^\infty \frac{da_2}{\Psi_{G\backslash e_1}\Psi_{G/e_1}} &= \int_0^\infty \frac{da_2}{(\Psi^{12}a_2 + \Psi^1_2)(\Psi^2_1 a_2 + \Psi_{12})} \\ &= \frac{\log\left(\frac{\Psi^{12}a_2+\Psi^1_2}{\Psi^2_1 a_2 \Psi_{12}}\right)\Big|_{a_2=0}^{\infty}}{\Psi^{12}\Psi_{12} - \Psi^1_2\Psi^2_1} \\ &= \frac{\log\left(\frac{\Psi^{12}\Psi_{12}}{\Psi^1_2\Psi^2_1}\right)}{\Psi^{12}\Psi_{12} - \Psi^1_2\Psi^2_1}.\end{aligned}$$

Now all of these Ψ can be viewed as determinants. In particular consider the matrix

$$M = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix},$$

where Λ is the diagonal matrix of the a_i and E is a signed incidence matrix of the graph with one row removed. Then by the matrix tree theorem $\det M = \Psi$. Now let's use the notation $\Psi^{I,J}_K = \det M(I,J)|_{a_e=0, e\in K}$ where $M(I,J)$ is the matrix M with rows indexed by I removed and columns indexed by J removed. Note that this is compatible with the notation defined earlier this lecture with the observation that $\Psi^I = \Psi^{I,I}$.

With this notation the classical Dodgson identity is $\det M \det M(12,12) = \det M(1,1)\det M(2,2) - \det M(1,2)\det M(2,1)$. Applying this to our M and setting $a_1 = 0$, $a_2 = 0$ we get that the denominator in the last expression above is equal to $(\Psi^{1,2})^2$.

We can keep going with exact statements if we know some special functions, but let's just sketch it. So far we had

$$\int \frac{1}{\Psi^2} = \int \frac{1}{\Psi^1\Psi_1} = \int \frac{\text{logs}}{(\Psi^{1,2})^2}.$$

Continuing, we get

$$\begin{aligned}
&= \int \frac{\text{more logs}}{\Psi^{13,23}\Psi^{1,2}_3} \\
&= \int \frac{\text{dilogs}}{\Psi^{12,34}\Psi^{13,24}} + \frac{\text{dilogs}}{\Psi^{12,34}\Psi^{14,23}} + \frac{\text{dilogs}}{\Psi^{13,24}\Psi^{14,23}} \\
&= \int \frac{\text{trilogs and other weight 3 stuff}}{\Psi^{12,34}_5\Psi^{135,245} - \Psi^{13,24}_5\Psi^{125,345}}
\end{aligned}$$

The denominator at this last step we'll call D_5. Each time the denominator is not a square then the denominator at the next step is the discriminant of the denominator at the previous step and the weight of the stuff in the numerator goes up by 1. D_5 does not always factor, but when it does this pattern continues, and so on.

Weight here is weight in the sense of the multiple zeta values and multiple polylogarithms we discussed last class. Intuitively weight is the minimum nested integral depth needed to obtain the quantity in question.

To proceed let's split into two algorithms. Just for the denominators there is a purely polynomial game to play. This is called *denominator reduction.*

Start with D_5. If $D_n = (Aa_i + B)(Ca_i + D)$, distinct factors, then $D_{n+1} = AD - BC$. If the factors are the same then $D_{n+1} = 0$. If D_n doesn't factor the algorithm ends.

This mirrors what happens to the denominators in the following integration algorithm (Brown's integration algorithm)

Start with the explicit expression involving trilogs and D_5. Then use the following identities for $L_w(z)$:

$$\int \frac{L_w(z)dz}{(z-\sigma_i)(z-\sigma_j)} = \frac{1}{\sigma_i - \sigma_j}(L_{x_iw}(z) - L_{x_jw}(z))$$

$$\int \frac{L_{x_i^r x_j w}(z)dz}{(z-\sigma_i)^2} = \frac{1}{\sigma_i - \sigma_j}(L_{x_iw}(z) - L_{x_jw}(z)) + \frac{1}{1-\sigma_i}\sum_{k=1}^{r}(-1)^{k+1}L_{x_i^{r-k}x_jw}(z)$$

for $i \neq j$.

Note how the first identity corresponds to the distinct factor case for the denominators and how the weight of the multiple polylogarithms goes up by 1 (after having done one integration, so this fits the intuition). In the second identity the weight does not go up even though we used an integration. Denominator reduction is only keeping track of the maximal weight piece and so that's why we get $D_{n+1} = 0$ in the corresponding situation.

On your assignment you'll see this in action on K_4.

Next time

Next time I will discuss weight drop and double triangles.

References

Francis' Brown's algorithms come from arXiv:0910.0114