COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 33 SUMMARY

WINTER 2018

Summary

We finished off our discussion of multiple zeta values and multiple polylogarithms today.

By how iterated integrals multiply, if w_1 and w_2 are words in x and y we have

$$\zeta(w_1 \sqcup\!\sqcup\, w_2) = \zeta(w_1)\zeta(w_2)$$

(we did an example in class; an example is quite illustrative).

What about the sum representation? That also shuffles but then there's an extra bit too. We did the example $\zeta(3)\zeta(2,2)$. Letting the sum indices be n and $m_1 > m_2$ respectively we have five cases $n > m_1 > m_2$, $m_1 > n > m_2$, $m_1 > m_2 > n$ and $n = m_1 > m_2$, $m_1 > n + m_2$. These become $\zeta(3,2,2)+\zeta(2,3,2)+\zeta(2,2,3)+\zeta(5,2)+\zeta(2,5)$. The first three are the shuffle part, the last two are the extra stuff. Because this is a shuffle with stuff it is often called the *stuffle*. It is usually notated $\star$.

This gives two ways to multiply multiple zeta values, and the two results typically look quite different. This gives many relations between multiple zeta values. With a small extension to some divergent words this is conjectured to give all relations between multiple zeta values.

Another way to generalize the Riemann zeta function is the *polylogarithm*

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

Note that $\mathrm{Li}_1(z) = -\log(1-z)$. $\mathrm{Li}_2(z)$ is called the *dilogarithm* and $\mathrm{Li}_3(z)$ is called the *trilogarithm*. The dilogarithm is perhaps the most important special function that you haven't heard about.

These have iterated integral representations just like the zeta values but with the outermost integral going from 0 to z.

We can also make a common generalization called *multiple polylogarithms*. Given a word w in x and y let $L_w(z)$ be the iterated integral as for $\zeta(w)$ but with the outermost integral going from 0 to z. In fact we don't need to restrict our letters to x and y. Given distinct complex numbers $\sigma_1, \ldots, \sigma_n$, let x_i correspond to $\frac{dt}{t-\sigma_i}$. Then we can use these differential forms to build iterated integrals from a word and build multiple polylogarithms in exactly the same way.

Next time

Next time I will discuss Brown's denominator reduction algorithm.

References

For multiple zeta values, here are some lecture notes by Michael Hoffman `people.math.sfu.ca/~summerschool/summer_school_2014/mzvcrs1.pdf`.

For the multiple polylogarithms see arXiv:0910.0114