COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 32 SUMMARY

WINTER 2018

Summary

We started by finishing the transition from momentum space to parametric space. We had ended off last time with

$$e^c\pi^{4\ell}\int_0^\infty\cdots\int_0^\infty da_1\cdots da_m\frac{e^{B^TA^{-1}B}}{(\det A)^2}$$

here A is the dual graph Laplacian with variables. That is the i,jth entry of A for $i\neq j$ will contain $-a_e$ if the momenta through the ith and jth cycles in our basis of cycles both contribute to e, that is if e is in both cycles. The diagonal entry i,i is the sum of the a_e for e in the ith cycle. This is like the Laplacian but with cycles in place of vertices (and with variables, but you can put variables in the Laplacian too).

Consequently, $\det A = \sum_T\prod_{e\notin T}a_e$ where the sum is over spanning trees of the graph. We will use the notation $\Psi=\det A$. For the $B^TA^{-1}B$ part see assignment 5.

Now we would still expect that this would diverge and Ψ is homogeneous which we haven't made use of yet. In particular $\Psi(\lambda a_1,\ldots,\lambda a_m)=\lambda^\ell\Psi(a_1,\ldots,a_m)$. Also we have $1=\int_0^\infty d\lambda\delta(\lambda-\sum_{e\in E(G)}a_e)$. So we can rescale the whole integral by $a_e\mapsto\lambda a_e$ to get

$$\begin{aligned}
&e^c\pi^{4\ell}\int_0^\infty\cdots\int_0^\infty d(\lambda a_1)\cdots d(\lambda a_m)\int_0^\infty d\lambda\delta(\lambda-\sum\lambda a_e)\frac{e^{\lambda B^TA^{-1}B}}{\Psi^2(\lambda a_1,\ldots,\lambda a_m)}\\
&=e^c\pi^{4\ell}\int_0^\infty\cdots\int_0^\infty da_1\cdots da_m\frac{\delta(1-\sum a_e)}{\Psi^2(a_1,\ldots,a_m)}\int_0^\infty d\lambda e^{\lambda B^TA^{-1}B}\lambda^{|E(G)|-2\ell}\\
&=(-1)^{|E(G)|-2\ell}e^c\pi^{4\ell}\Gamma(2\ell-|E(G)|)\int_0^\infty\cdots\int_0^\infty da_1\cdots da_m\frac{\delta(1-\sum a_e)}{\Psi^2(B^TA^{-1}B)^{|E(G)|-2\ell}}
\end{aligned}$$

Now for a ϕ^4 graph with 4 external edges $|E(G)|=2\ell$ so the Γ-function has a pole and the coefficient of this pole (ie residue) is

$$P_G=\int_{a_e\geq 0}da_1\cdots da_m\frac{\delta(1-\sum a_e)}{\Psi^2}$$

which we call the Feynman period of the graph. Note that we could have taken any linear function of the a_e in place of $\sum a_e$ in the delta function, so in particular we could just set $a_m=1$ and only integrate the rest of them. That will be easiest on your assignment.

The next thing we did was define *multiple zeta values* (MZVs) and two ways to represent them. The definition we gave was

$$\zeta(s_1,s_2,\ldots,s_k)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

where $s_1 + s_2 + \cdots + s_k$ is the weight of the MZV and k is the depth. MZVs also have an iterated integral representation. Let the letter x represent dt/t and let the letter y represent $dt/(1-t)$ and then iterate them as they appear in a word like in the following example

$$x^{n-1}y \mapsto \int_0^1 \frac{dt_n}{t_n} \int_0^{t_n} \frac{dt_{n-1}}{t_{n-1}} \cdots \int_0^{t_3} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1}$$

By expanding the geometric series and working up through the variables you can compute that this is $\zeta(n)$. In general the integration variables go from 0 to the previous variable, and a letter contributes its associated differential form in t_k if it is in position $|w| - k$ in the word w. We will also call this map ζ and so

$$\zeta(x^{s_1-1}yx^{s_2-1}y \cdots x^{s_k-1}y) = \zeta(s_1, s_2, \ldots, s_k)$$

Next time

Next time I will say more about how multiple zeta values relate to words and define multiple polylogarithms.

References

For the parametric Feynman integral calculations see Itzykson and Zuber "Quantum Field Theory" section 6-2-3 (the calculation is exactly there just with different letters).

For multiple zeta values, here are some lecture notes by Michael Hoffman `people.math.sfu.ca/~summerschool/summer_school_2014/mzvcrs1.pdf`