COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 2 SUMMARY

WINTER 2018

Summary

Today began our first topic of graph counting by integration. Our running example for the next week or so will be

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq e^{(-\frac{1}{2}q^2 + \frac{\lambda}{3!}q^3 + Jq)}$$

As a first step to analysing this we went over the definitions for formal power series. You can find details in any rigorous introduction to combinatorial enumeration. I don't have a particular favorite reference for formal power series, so if some of you do you can mention it and we can include it here for others. The only thing in the way I set things up which is not completely standard is the definition of *valid*:

Definition 1. *Suppose $D \subseteq R[[x]]^k$ and $\Theta : R \to R[[x]]$, then we say Θ is* valid *if for all $n \in \mathbb{Z}_{\geq 0}$ there exists an $m \in \mathbb{Z}_{\geq 0}$ such that for any input series $F_i(x) = \sum_{j \geq 0} f_{i,j} x^j$ with $(F_1, \ldots, F_k) \in D$, the coefficient*

$$[x^n]\Theta(F_1(x), \ldots, F_k(x))$$

depends only on $f_{i,j}$ with $0 \leq i \leq k$ and $0 \leq j \leq m$.

The reason to define this is that we don't need to go into questions of formal power series topology and convergence therein in order to understand what makes sense as a formal power series operation. The downside is that the definition is more ad-hoc and less powerful.

The other thing of note, is that I will use the notational convention that formal power series whose names are upper case letters have the corresponding lower case letters as their coefficients, unless otherwise specified. For example $A(x) = \sum_{n \geq 0} a_n x^n$.

With formal power series as a tool we can expand our running example to obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq e^{(-\frac{1}{2}q^2 + \frac{\lambda}{3!}q^3 + Jq)} = \sum_{\substack{i \geq 0 \\ j \geq 0}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}q^2} q^{j+3i} \right) \frac{J^j}{j!} \frac{\lambda^i}{(3!)^i i!}$$

The next step is to find a combinatorial way to understand $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}q^2} q^{j+3i}$ (and this will give the promised connection to counting graphs).

To do that we need to understand Gaussian integrals. We finished off today's class with the calculation

Proposition 2.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} = 1$$

The proof is elementary calculus; the trick is to start by squaring it. You can find this proof in many sources because it is quite standard. For example it is example 3.2.1 of Lando and Zvonkin.

The point is that $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$ defines a normalized measure on $\mathbb{R}$ called the *Gaussian measure*

Next time

Next class we will continue investigating the Gaussian measure on $\mathbb{R}$ in order to evaluate our running example in terms of graphs.

References

"Graphs on surfaces and their applications" by Lando and Zvonkin (Springer 2004).