COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 29 SUMMARY – PART 2

WINTER 2018

Summary

In the second part of today's lecture we began the new topic of parametric Feynman integration.

The first step is to see how to move from momentum space (where the Feynman integrals we have seen so for are) to parametric space. We need two integral facts:

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^tAx+Bx+c}dx_1\cdots dx_n = \sqrt{\frac{(2\pi)^n}{\det A}}e^{\frac{1}{2}B^TA^{-1}B}e^c$$

$$\frac{1}{p^2} = \int_0^{\infty} e^{-ap^2}da.$$

The first is one of the things we talked about before regarding Gaussian integrals (with an extra e^c) and the second is a calculus exercise.

Now we want to apply the second fact to each edge in the momentum space Feynman integral. Set it up like this. Let G be the graph, let q_i be the external momenta. Let $C_1, C_2, \dots, C_\ell$ be an oriented basis for the cycle space of G and let $k_1, k_2, \dots, k_\ell$ be the associated momenta. Let p_e be the momentum through edge e (this will be a signed sum of the k_i depending on which cycles go through e in which direction along with external momenta). We'll restrict to the massless case though something similar holds with mass. The Feynman integral for G in momentum space is (this is the version where you used the delta functions to reduce to momenta only for the cycles):

$$\begin{aligned}
&\int d^4k_1\cdots d^4k_\ell \prod_{e\in E(G)}\frac{1}{p_e^2}\\
&= \int d^4k_1\cdots d^4k_\ell \prod_{e\in E(G)}\int_0^{\infty} e^{-a_ep_e^2}da_e\\
&= \int_0^{\infty}\cdots\int_0^{\infty} da_1\cdots da_m \int d^4k_1\cdots d^4k_\ell e^{-\overline{k}^tA\overline{k}+B\overline{k}+c}\\
&= e^c\pi^{4\ell}\int_0^{\infty}\cdots\int_0^{\infty} da_1\cdots da_m\frac{e^{B^TA^{-1}B}}{(\det A)^2}
\end{aligned}$$

where $\overline{k}$ is the vector of the k_i. Note that we get $(\det A)^2$ in the end rather than $\sqrt{\det A}$ because the k_i are all 4-vectors, so we have 4 times the regular multivariate Gaussian, hence $\left(\sqrt{\det A}\right)^4$. The final question is what are the matrices?

Consider A. The i, jth entry of A will contain $-a_e$ if k_i and k_j are both in the momentum of e. This means that A is just like the graph Laplacian (with variables) but with cycles instead of vertices; so it is the dual Laplacian. Fortunately this means we know about the determinant of A from the matrix-tree theorem. This is what Iain will discuss next time.

Next time

Iain Crump will tell you about the matrix tree theorem.

References

See section 6-2-3 of Itzykson and Zuber "Quantum Field Theory" (there's a cheap Dover edition if you want a paper copy).