COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 26 SUMMARY

WINTER 2018

Summary

Last class we'd looked at the QED system (see p47 of the reference after lecture 19 for the system itself). Then we observed that we could rewrite this system in the form

$$X^r = 1 \pm \sum_k B_+^{\gamma_{r,k}}(X^r Q^k)$$

where $\gamma_{r,k}$ is the sum of primitives of loop order k and with external structure r and $Q = (X^{\text{vertex}})^2/((X^{\text{fermion}})^2 X^{\text{photon}})$

Call this Q the *combinatorial invariant charge.*

Proposition 1. *Let T be a combinatorial physical theory with one vertex type v. let $d = deg(v)$, let $n(e)$ be the number of half edges of type e making up vertex v (count either half for directed edges) divided by 2, and $n(v) = 1$. Let G be a connected graph in this theory with loop order ℓ and external leg structure r one of the vertices or edges in the theory. Then G has*

- $\frac{2\ell n(s)}{d-2}$ *insertion places of every type $s \neq r$ and*
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  - *if $r = v$ then G has $\frac{2\ell n(r)}{d-2} + 1$ insertion places of type r while*
  - *if $r \neq v$ then G has $\frac{2\ell n(r)}{d-2} - 1$ insertion places of type r.*

Therefore

$$Q = \left(\frac{X^v}{\prod_e (X^e)^{n(e)}}\right)^{\frac{2}{d-2}}$$

Note that the $n(e)$ are half of what we said in class. This is the correct version. The proof of the proposition is by counting; we went over it in class.

Combinatorial specifications built with B_+ in either $\mathcal{H}$ or any $\mathcal{H}_T$ we will call *combinatorial Dyson-Schwinger equations.*

Solutions to combinatorial Dyson-Schwinger equations which come from physics tend to give sub Hopf algebras. The reason this makes sense is that typically a solution $T(x)$ will give something like a sum of all graphs with a given external leg structure. The point of renormalizability was to be able to deal with all these at once. So in order to apply renormalization, which we know to be essentially Hopf algebraic, we need $T(x)$ to be well behaved with respect to the coproduct.

Here's the specific set up for a nice characterization result of Loïc Foissy.

Consider

$$T(x) = xB_+(f(T(x)) \tag{1}$$

as an equation for an augmented generating fucntion in $\mathcal{H}$ with f a formal power series with $f(0) = 1$. Observe that $T(x) = \sum_{n \geq 1} t_n x^n$ is uniquely defined by the validity of these formal power series operations. Now consider $\mathbb{Q}[t_1, t_2, \ldots]$. This is certainly a sub algebra of $\mathcal{H}$ as an algebra, but it may or may not be a sub Hopf algebra. As usual the counit is automatic, so the question is whether or not $\Delta(t_i) \in \mathbb{Q}[t_1, t_2, \ldots] \otimes \mathbb{Q}[t_1, t_2, \ldots]$. If so then we say that the Dyson-Schwinger equation (1) is *Hopf*.

Next time

Next class we'll state and sketch a proof of Foissy's result.

References

For insertion see section 5.4 of the reference linked beside lecture 19.
Foissy's paper is arXiv:0707.1204