COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 24 SUMMARY

WINTER 2018

Summary

Today we finished our discussion of renormalization and just got a little start of the new section on Dyson-Schwinger equations.

The remaining thing to discuss for renormalization was the map R, which is called the *renormalization scheme*. In the formal integral world, R would be, like in the toy model, the map where you evaluate at fixed values of the external parameters. If you are using dimensional regularization (a regularization scheme where you view the integral as over a space of dimension $D-2\epsilon$ and expand in ϵ) then you can renormalize by *minimal subtraction* in which case R is the map $R(\sum_{i\geq I} f_i\epsilon^i) = \sum_{I\leq i<0} f_i\epsilon^i$. The mathematical formulation of renormalization only requires that R is a Rota-Baxter map:

Definition 1. *For an algebra A over a field K of characteristic 0, a linear map $R : A \to A$ is a Rota-Baxter operator if*

$$R(a)R(b) = R(R(a)b + aR(b) + \theta ab)$$

for some fixed $\theta \in K$ and all $a, b \in A$.

Minimal subtraction and evaluating formal integrals are both Rota-Baxter.

With all this then as for the toy we have

$$S_R^F(G) = -R(F(G)) - \sum_{\substack{1\neq\gamma\subsetneq G \\ \gamma \text{product of} \\ \text{divergent 1PI}}} S_R^F(\gamma)R(F(G/\gamma))$$

$$F_{\text{ren}} = S_R^F \star F$$

defining the renormalized Feynman rules F_{ren}.

You can view this as an example of Birkhoff decomposition (see the original papers of Connes and Kreimer). On this physics side this is an algebraic formulation of what is known as BPHZ renormalization.

To finish the day we went back to combinatorial specifications and rewrote things with B_+, for example

$$B(x) = xB_+(1) + xB_+(B(x)^2)$$

as an equation for the augmented generating function of binary rooted trees.

Now we want to do something similar for graphs. The first step is to define the *insertion* of G_1 into G_2, $G_1 \circ G_2$. This is defined to be the sum over all possible ways to insert G_1 into G_2, where a way to insert is as follows. If the external leg structure of G_1 is an edge of the theory then you can insert G_1 into any internal edge of that type in G_2 (one way for an oriented edge type and two ways for an unoriented type) and if the external leg structure of

G_1 is a vertex of the theory then you can insert G_1 into any vertex of that type in G_2, once for each type-preserving bijection of the external half edges of G_1 with the half edges of the vertex.

We ended with the statement that this $\circ$ is a *pre Lie product*. We'll return to that fact and to examples next time.

Next time

Next class we'll continute thinking about graph insertion.

References

Lectures by Dominique Manchon on renormalization Hopf algebras arXiv:math/0408405.

A review by Ebrahimi-Fard and Kreimer with some emphasis on the Rota-Baxter operator: arXiv:hep-th/0510202

One of the original Connes and Kreimer papers: arXiv:hep-th/9912092