COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 23 SUMMARY

WINTER 2018

Summary

Today we looked at how to apply the theorem from last class to the situation of QFT and started looking at how to renormalize with graph Hopf algebras

First note that we had

$$\psi[K]_a = \frac{\partial T[K]}{\partial K_a}$$

which defines the ψ as formal power series in the K. However this can be inverted by the multivariate Lagrange theorem. It turns out that the inverse relation is

$$K[\psi]_a = -\frac{\partial F[\psi]}{\partial \psi_a}$$

which can be obtained by differentiating the expression for $F[\psi]$ from the theorem last time and using the expression for $\psi[K]_a$. From this you also see how the Legendre transform is close to being an involution; there's just a sign modifying things in reverse.

Now let's think about how this transfers to the QFT situation. We want J to play the role of K and ϕ to play the role of ψ (or almost, there is a caveat as we'll see).

Then $T[K]$ corresponds to $W[J]$ where $W[J]$ is viewed as a generating function of trees where the edges of each tree are weighted by the propagators and the vertices of each tree are the 1PI amplitudes (that is the sum of the 1PI graphs with the appropriate external structure for that vertex). Thus $F[\psi]$ is the generating functional for the 1PI diagrams, namely the effective action $\Gamma[\phi_{\mathrm{cl}}]$. So the theorem says that given $W[J]$ as we already had it defined, if we define $\Gamma[\phi_{\mathrm{cl}}]$ according to the relation defined in theorem then $\Gamma[\phi_{\mathrm{cl}}]$ really will be the 1PI generating functional. (The theorem literally says that if Γ and W are defined to be what we combinatorially want them to be then they satisfy the relation, but the relation defines either one of them uniquely from the other, so we can use theorem to define Γ from W.)

There are two things to note in this translation. First ϕ and J are fields so they comes from an infinite dimensional function space, but above we had a finite number of ψ and K. To make the number of variables finite we would have had to truncate our function space in some way so as to make it finite dimensional, let the ψ_a and K_a be a basis and then take a limit at the end to return us to our actually infinite dimensional space. In a given circumstance this may or may not make rigorous sense. It would be nice to have a true formal theory of these kinds of expansions, but then we would need some space of formal integrals. To see why this is necessary, consider the defining equation of the Legendre transform, translate to QFT and infinite dimensions. It becomes

$$\Gamma[\phi_{\rm cl}] = W[J] - \int d^4x \phi_{\rm cl} J$$

Note that the integral which appears is just over $\mathbb{R}^4$ not a functional integral, so it is a well-defined object analytically, the question is how make it play well with the formal power series.

The second thing to note concerns $\phi_{\rm cl}$. This is the classical field. It is called that because

$$\phi_{\rm cl} = \frac{\delta W[J]}{\delta J}.$$

(It is usual notation to switch to δ here since we are now taking a derivative with respect to a field.) Then by the theorem itself, this $\phi_{\rm cl}$ is the solution to

$$\frac{\delta}{\delta J}(\int d^4k \phi_{\rm cl} J - W[J]) = 0$$

so it is minimizing something. The classical version of something is always the one that minimizes the defining equations.

Now back to the renormalization. The point of that discussion of the Legendre transform was to justify why we have built our Hopf algebras with 1PI graphs. We also mentioned before that the graph Hopf algebras line up with the Connes-Kreimer Hopf algebra by taking the insertion tree of a graph (the tree given by containment of divergent 1PI subgraphs). However, some graphs have overlapping subdivergences. You can still deal with this in the Connes-Kreimer Hopf algebra by taking a sum of the different insertion trees which can give the graph. None the less we want to avoid this by phrasing renormalization directly on the renormalization Hopf algebras of graphs.

The formula ends up being essentially the same (for S_R^F and $F_{\rm ren}$), but we do still need to think about what the various maps are. First of all here F is the Feynman rules. We want to think about $F : \mathcal{H}_T \to A$ where T is the combinatorial physical theory and A is some algebra. What algebra is it? The standard answer is to regularize your integrals in some way so that they become Laurent series in some new parameter ϵ with coefficients that are functions of the remaining parameters. One could also, as with the toy models, have A be some space of formal integrals. The next step is to consider what R can be, which we'll discuss next time.

Next time

Next class we'll discuss what renormalization schemes are and hopefully begin Dyson-Schwinger equations.

References

Jackson, Kempf, Morales, "A robust generalization of the Legendre transform for QFT". arXiv:1612.00462

Lectures by Dominique Manchon on renormalization Hopf algebras arXiv:math/0408405.