COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 22 SUMMARY

WINTER 2018

Summary

Today we discussed the combinatorial Legendre transform.

Recall the generating functional $Z[J]$ and how we can see $Z[J]$ as resulting from applying Feynman rules divided by the size of the automorphism group to the augmented generating function of Feynman graphs. As we noted before dividing by the size of the automorphism group means this is secretly all labelled, so $\log(Z[J])$ is the analogous sum over connected diagrams. The usual notation is

$$W[J] = \log(Z[J])$$

which is the generating functional for the *connected* diagrams. There's still more we can do because any connected graph is a tree of its 1PI pieces. Note that the bridges connecting the 1PI pieces aren't involved in the integrals of the Feynman integral; they just contribute a factor independent of the integration variables.

The analogous sum over 1PI diagrams is called the *effective action* and is written

$$\Gamma[\phi]$$

The way to get from $W[J]$ to $\Gamma[\phi]$ is by the Legendre transform. What we did today is prove a version of the Legendre transform for formal power series by Jackson, Kempf, and Morales. The proof comes down to the simple fact that $|V(T)| - |E(T)| = 1$ for any tree T. Here is the theorem:

Theorem 1. *Let $\psi_1, \ldots, \psi_N$ and $K_1, \ldots, K_N$ be indeterminates.*

Let $F[\psi]$ be a formal power series in the ψ_i. Write $F^{(n)}_{a_1,\ldots,a_n}/n!$ for the coefficient of $\psi_{a_1} \cdots \psi_{a_n}$ in $F[\psi]$. Assume $F[\psi]$ has no constant or linear terms. Assume also that the only nonzero quadratic coefficients are the $F^{(2)}_{a,a}$ and that these are all invertible.

Let $T[K]$ be the generating function for trees with edges coloured by the index set $\{1, \ldots, N\}$ and with a weight which is a product of the following factors. (Note that these trees do not have external edges in the sense of dangling half edges, rather there can be 1-valent vertices.) Vertices of degree > 1 are weighted by a factor of $F^{(d)}_{a_1,\ldots a_d}$ where d is the degree and $a_1, \ldots a_d$ are the colours of the incident edges. 1-valent vertices are weighted by K_a where a is the label of the incident edge. An edge coloured a is weighted by $-(F^{(2)}_{a,a})^{-1}$.

Define a relationship between K and ψ by

$$\psi[K]_a = \frac{\partial T[K]}{\partial K_a}.$$

Then

$$T[K] = \sum_{a=1}^{N} K_a \psi[K]_a + F[\psi[K]]$$

Note that this last equation is the defining equation of the Legendre transform, so this is saying that T and F relate by a Legendre transform.

Proof. Take a monomial $K_{a_1} \dots K_{a_m}$ and consider the coefficient on each side.

$$\frac{\partial}{\partial K_{a_1}} \cdots \frac{\partial}{\partial K_{a_m}} T[K]$$

is the sum of trees (weighted as above) with 1-valent vertices labelled $K_{a_1}, \dots, K_{a_m}$.

Now consider

$$\frac{\partial}{\partial K_{a_1}} \cdots \frac{\partial}{\partial K_{a_m}} (\sum K_a \psi[K] + F[\psi[K]])$$

the claim is that this counts the same trees with each tree T counted exactly $|V(T)| - |E(T)|$ times.

Observe $K_b \psi[K]_b = K_b \frac{\partial T[K]}{\partial K_b}$ which is the sum of all trees with one distinguished 1-valent vertex K_b. Note that the K_b factor for this vertex has been put back in.

Likewise in $F[\psi[K]]$ there are terms of the form

$$\frac{1}{2} F^{(2)}_{b,b} \psi[K]_b \psi[K]_b$$

which is a sum of all pairs of two trees each with a distinguished K_b 1-valent vertex. Join these two 1-valent vertices together and pop off the subdivision so that we just have a single b edge in this location. Note that the edges incident to the distinguished 1-valent vertices in the pair of trees each contributed a factor of $-(F^{(2)}_{b,b})^{-1}$, but we only want one such factor now. Fortunately the extra ones cancels with the explicit $F^{(2)}_{b,b}$ above. The $1/2$ takes care of the permutation symmetry between the two trees of the pair. The result then is that this term sums over trees of the kind we are interested in with a distinguished b-edge.

All the other terms in $F[\psi[K]]$ are of the form

$$\frac{1}{n!} F^{(n)}_{b_1,\dots,b_n} \psi[K]_{b_1} \cdots \psi[K]_{b_n}$$

which is a sum of trees, one with a distinguished K_{b_1} 1-valent vertex, one with a distinguished K_{b_2} 1-valent vertex and so on. Join all these distinguished vertices into a new n-valent vertex. The only thing missing is the $F^{(n)}_{b_1,\dots,b_n}$ which is explicit in the above. Again the $1/n!$ takes care of the permutations of the original n trees.

From these observations we see that

$$\frac{\partial}{\partial K_{a_1}} \cdots \frac{\partial}{\partial K_{a_m}} (\sum K_a \psi[K] + F[\psi[K]])$$

counts the trees with 1-valent vertices labelled $K_{a_1}, \dots, K_{a_m}$ and with a distingusihed vertex (of any valence) or a distinguished edge, and the ones with the distinguished edge appear with a negative sign. Thus each tree T appears $|V(T)| - |E(T)| = 1$ times. The result follows. □

Next time

Next class we'll apply this theorem to the QFT case and talk about how to renormalize with renormalization Hopf algebras.

References

Take a look at Jackson, Kempf, Morales, "A robust generalization of the Legendre transform for QFT". arXiv:1612.00462