COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 21 SUMMARY

WINTER 2018

Summary

Today we defined renormalization Hopf algebras of Feynman graphs.

Definition 1. *A connected Feynman graph is* one particle irreducible *or* 1PI *if after removing any one internal edge it remains connected. This is 2-edge-connected for graph theorists.*

We'll see next class why we can restrict to 1PI graphs, but first let's use them so that you have what you need for your assignment.

Fix a combinatorial physical theory T. Let $\mathcal{G}$ be the set of divergent 1PI graphs in the theory (sometimes we don't want divergent here, but let's stick to divergent for now). The Hopf algebra $\mathcal{H}_T = \mathbb{Q}[\mathcal{G}]$ as an algebra. As usual identify product with disjoint union so the elements of $\mathcal{H}_T$ are graphs in the theory T whose connected components are 1PI.

Grade $\mathcal{H}_T$ by loop order (i.e. dimension of cycle space). However, there's a problem $\mathcal{H}_T$ is not connected (and $\mathsf{MSet}(\mathcal{G})$ is not a combinatorial class with this size function) because there can be many graphs of size 0, all the vertices of the theory.

To fix this note that in the end what we really want is sums of the Feynman integrals of graphs with the same external leg structure. Thus it won't hurt to scale all these graphs by some constant. In particular we can scale so that the vertex itself contributes 1. Then we can also identify these on the combinatorial side. Specifically, remove the vertices from $\mathcal{G}$ and identify all the vertices with $1 \in \mathcal{H}_T$.

With this, $\mathcal{H}_T$ is a graded connected algebra.

To define the coproduct we first need to understand the correct definition of subgraph and contraction of subgraphs.

Definition 2. *A subgraph γ of a Feynman graph G is a subset of the half edges of G that is full at each vertex (that is if one half edge at a vertex v of G is in γ then all half edges at that vertex are in γ and v is also a vertex of γ), but the edge relation can be any subset of the edge relation of G restricted to the half edges of γ.*

This is the right definition of subgraph because the vertices came from the path integral in a way which was indivisible but the edge pairings you can take or not take.

Given a connected subgraph γ of G. If γ has external leg structure a vertex of T then to contract γ just replace γ by this vertex. If γ has external leg structure an edge of T then to contract γ remove γ and then rejoin the two new dangling edges into an internal edge. To contract a disconnected γ contract each connected component.

Now, suppose T is renormalizable and all the divergent external leg structures are edges or vertices of the theory (this is what actual physical renormalization ends up forcing). Then define

$$\Delta(G) = \sum_{\substack{\gamma \subseteq G \\ \gamma \text{ product of} \\ \text{divergent 1PI} \\ \text{subgraphs}}} \gamma \otimes G/\gamma$$

and extend as an algebra homomorphism. Take our usual counit. This then defines $\mathcal{H}_T$ as a Hopf algebra.

Note that the connection to the Connes-Kreimer Hopf algebra is via insertion trees. We'll get back to that in a couple of classes.

Next time

Next class we'll talk about how to use the Legendre transform to restrict to 1PI.

References

Here are some lectures by Dominique Manchon that you might like `https://arxiv.org/abs/math/0408405`