COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 19 SUMMARY

WINTER 2018

Summary

Today we defined the Connes-Kreimer Hopf algebra.

Let $\mathcal{T}$ be the combinatorial class of rooted trees with no plane structure and no empty tree (yet). Then let $\mathcal{H} = \mathbb{Q}[\mathcal{T}]$ identifying multiplication with disjoint union so we can see this as an algebra of forests. Note that the empty tree/forest has returned as the empty monomial 1. This is the algebra structure of the Connes-Kreimer Hopf algebra. Next for the coalgebra structure, for $t \in \mathcal{T}$ define

$$\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ C \text{ antichain}}} \left(\prod_{v \in C} t_v\right) \otimes \left(t - \prod_{v \in C} t_v\right)$$

and extended as an algebra homorphism, where t_v is the subtree rooted at v. Note that you should interpret $t - \prod_{v\in C} t_v$ as 1 if $\prod_{v \in C} t_v = t$ (this is a deficiency of my notation really as you should think of the removals on the right as being multiplicative in style, like contraction, even though as defined they are subtraction).

Often this coproduct is instead presented in terms of admissible cuts which are the edges immediately rootwards of the $v \in C$ the way I have defined it.

The counit is $\epsilon(1) = 1$ and $\epsilon(t) = 0$ for $t \in \mathcal{T}$. Note that this is graded and connected and so is a Hopf algebra. Specifically this is the *Connes-Kreimer Hopf algebra* which we will write $\mathcal{H}$ without further comment. We did some examples of both Δ and S.

Next we defined the add-a-root operator: $B_+(t_1 \cdots t_k)$ is the tree with a new root vertex and the subtrees at the root are $t_1, t_2, \ldots, t_k$. Observe that

$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes 1$$

This means that B_+ is a Hochschild 1-cocycle. We talked defined Hochschild cohomology and I gave a 3-point summary of cohomology in general, but we won't actually need anything beyond the 1-cocycles for this class.

We finished with the following key theorem.

Theorem 1. *Let A be a commutative algebra and $L : A \to A$ a map. Then there exists a unique algebra homomorphism $\rho_L : \mathcal{H} \to A$ such that $\rho_L \circ B_+ = L \circ \rho_L$. If further A is a bialgebra and L is a Hochschild 1-cocycle then ρ_L is a bialgebra homomorphism. If even further A is a Hopf algebra then ρ_L is a Hopf algebra homomorphism.*

We'll use this theorem in two ways, both important. First when A is a Hopf algebra then this is saying that for pairs of a Hopf algebra and a 1-cocycle the Connes-Kreimer Hopf algebra with B_+ is universal. Second, when A is the target algebra for Feynman rules then this theorem is explaining that if we know what B_+ should do after applying Feynman rules then we know the Feynman rules themselves (here ρ_L would be the Feynman rules).

Next time

Next time we'll use the Connes-Kreimer Hopf algebra to renormalize in a toy model.

References

Phrased like this, see sections 4.2 and 4.4 of my little book, "A combinatorial perspective on quantum field theory". There's a pdf linked on the course website. Also, see Erik Panzer's masters thesis arXiv:1202.3552.