COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 18 SUMMARY

WINTER 2018

Summary

Today we proved a few Hopf algebra properties that will be useful to us.

Proposition 1. *Let B be a graded connected bialgebra over K.*

(1) $u : K \to B_0$ *is an isomorphism.*
(2) $\epsilon|_{B_0} : B_0 \to K$ *is the inverse isomorphism.*
(3) $ker\epsilon = \bigoplus_{n=1}^{\infty} B_n$.
(4) *For* $x \in ker\epsilon$, $\Delta(x) = 1 \otimes x + x \otimes 1 + \widetilde{\Delta}(x)$ *where* $\widetilde{\Delta}(x) \in ker\epsilon \otimes ker\epsilon$.

The proofs can be found in Grinberg and Reiner.

If $\Delta(x) = 1 \otimes x + x \otimes 1$ then we say x is *primitive*. Another notion important for general Hopf algebras but not so much for us is that if $\Delta(x) = x \otimes x$ then we say x is group-like.

Proposition 2. (1) *Let H be a Hopf algebra. Then S is an antiautomorphism, that is* $S(1) = 1$ *and* $S(ab) = S(b)S(a)$.
(2) *Let H be a Hopf algebra. If H is commutative or cocommutative then* $S \circ S = id$.
(3) *Let B be a graded connected bialgebra. A has a unique antipode S making A into a Hopf algebra. Further S is graded so A is a graded Hopf algebra.*

The proofs can again be found in Grinberg and Reiner. The point for us is the last one because how it works is that $S \star \mathrm{id} = u\epsilon$ can be turned into a recurrence as follows: take $x \in \ker \epsilon$ and write $\widetilde{\Delta}(x) = \sum_{(x)} x_1 \otimes x_2$ (note the notation!) then

$$0 = u\epsilon(x) = (S \star \mathrm{id})(x) = x + S(x) + \sum_{(x)} S(x_1)x_2$$

so

$$S(x) = -x - \sum_{(x)} S(x_1)x_2$$

which recursively defines S (along with $S(1) = 1$) using the grading.

Next time

Next time we'll define the Connes-Kreimer Hopf algebra and derive some of its important properties.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356.