COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 17 SUMMARY

WINTER 2018

Summary

We started by defining the deconcatenation and deshuffle coproducts on words:

$$\Delta_{\mathrm{ds}}(a_1 a_2 \cdots a_n) = \sum_{i=0}^{n} a_1 \cdots a_i \otimes a_{i+1} \cdots a_n$$

$$\Delta_{\mathrm{dc}}(a_1 a_2 \cdots a_n) = \sum_{I \subseteq \{1,2,\ldots,n\}} a_I \otimes a_{\{1,2,\ldots,n\}-I}$$

where $a_I = a_{i_1} a_{i_2} \cdots a_{i_k}$ where $I = \{i_1 < i_2 < \cdots < i_k\}$. Then by trying some examples you saw that the shuffle product and the deshuffle coproduct don't go together to make a bialgebra and neither do the concatenation prpoduct and deconcatenation coproduct. The other two possibilities do work and give us our first examples of bialgebras.

Definition 1. *Let C be a coalgebra and A an algebra. Let $f, g : C \to A$ be linear maps. The* convolution product *of f and g is*

$$f \star g = m \circ (f \otimes g) \circ \Delta$$

Note that the $\star$-identity is $u\epsilon$.

Definition 2. *A bialgebra B is a* Hopf algebra *if there exists a convolution inverse of the identity map. That is if there exists a linear map $S : B \to B$ called the* antipode *such that*

$$S \star id = id \star S = u\epsilon$$

Definition 3. *A vector space V is* graded *(properly $\mathbb{Z}_{\geq 0}$-graded) if*

$$V = \bigoplus_{n=1}^{\infty} V_n$$

a map $f : V \to W$ is graded *if $f(V_n) \subseteq W_n$. An algebra, coalgebra, bialgebra, Hopf algebra, etc is* graded *if the underlying vector space and all the defining maps are graded.*

Both word bialgebras are graded.

Definition 4. *A graded vector space over K is connected if $V_0 \cong K$*

Both word bialgebras are connected.

We will often work with example that go like this: $\mathcal{C}$ is a combinatorial class with no elements of size 0 (typically it consists of connected objects in some sense). Then we can identify $K[\mathcal{C}]$ with $\mathrm{Span}_K(\mathsf{MSet}(\mathcal{C}))$ by idenfitying monomials with disjoint unions. This is a connected graded algebra. The interesting thing is to find a nice coproduct. We'll see next time that the antipode comes for free.

Next time

Next time we'll prove some propositions in order to get the basic Hopf algebra properties we need, most notably that the antipode is free in the graded connected case.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356.