COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 16 SUMMARY

WINTER 2018

Summary

After beginning the class with a bit of discussion of renormalizability including checking that quantum electrodynamics is renormalizable in the combinatorial sense we got on to the definition of Hopf algebras.

The idea is that a product of combinatorial objects takes two objects and returns one, or multiple ways to combine them into one. Two good examples are the concatentation product of words and the shuffle product of words. The shuffle product returns not just one word but all shuffles which we will capture by a formal sum. Specifically

$$\begin{aligned} aw_1 \sqcup\!\sqcup\, bw_2 &= a(w_1 \sqcup\!\sqcup\, bw_2) + b(aw_1 \sqcup\!\sqcup\, w_2) \\ 1 \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, 1 &= w \end{aligned}$$

where 1 is the empty word.

Then a coproduct is the opposite of this: a map that breaks an object into two pieces in one or more ways. To define all this and the compatibilities we need the easiest way is to use commutative diagrams.

Definition 1. *An algebra A over K is a vector space over K with two linear maps $m : A \otimes A \to A$, called the product or multiplication, and $u : K \to A$, called the unit, such that the following two diagrams commute*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes m} & A \otimes A \\ \Big\downarrow{\scriptstyle m \otimes id} & & \Big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

$$\begin{array}{ccccc} K \otimes A & \xleftarrow{a \mapsto 1 \otimes a} & A & \xrightarrow{a \mapsto a \otimes 1} & A \otimes K \\ \Big\downarrow{\scriptstyle u \otimes id} & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle id \otimes u} \\ A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \end{array}$$

To get a coalgebra we just reverse the arrows.

Definition 2. *A coalgebra C over K is a vector space over K with two linear maps $\Delta : C \to C \otimes C$, called the coproduct, and $\epsilon : C \to K$, called the counit, such that the following two diagrams commute*

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{id \otimes \Delta} & C \otimes C \\ \Big\uparrow{\scriptstyle \Delta \otimes id} & & \Big\uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow[1]{\Delta} & C \end{array}$$

$$
\begin{array}{ccccc}
K \otimes C & \xrightarrow{k\otimes c \mapsto kc} & C & \xleftarrow{c\otimes k \mapsto kc} & C \otimes K \\
\uparrow{\scriptstyle \epsilon\otimes id} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id\otimes\epsilon} \\
C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C
\end{array}
$$

We can also write the notion of algebra homomorphism in this language and then get the notion of coalgebra homomorphism immediately by reversing the arrows.

Definition 3. *Let A and B be algebras over K. A linear map $f : A \to B$ is an algebra homomorphism if the following diagrams commute*

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle m_A}\uparrow & & {\scriptstyle m_B}\uparrow \\
A \otimes A & \xrightarrow{f\otimes f} & B \otimes B
\end{array}
$$

$$
\begin{array}{ccc}
 & K & \\
{\scriptstyle u_A}\swarrow & & \searrow{\scriptstyle u_B} \\
A & \xrightarrow{f} & B
\end{array}
$$

Now suppose you have a vector space which is simultaneously an algebra and a coalgebra. Then the property that the two algebra maps are coalgebra homomorphisms is equivalent to the property that the two coalgebra maps are algebra homorphisms. The proof is just to write out the diagrams in each case and notice that they are the same. Such a vector space is called a *bialgebra*.

Next time

Next time we'll actually define Hopf algebras and state some examples and properties.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356. We will only need some intitial definitions and results from there, but it is a nice book.