COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 15 SUMMARY

WINTER 2018

Summary

We started with two examples of writing down the Feynman integral of a scalar field theory graph following the rules from last time. Note that we have set things up so that we only write down factors for the internal edges. (One says: the external edges are amputated.)

Today's main topic is divergences. If a Feynman integral diverges as the momenta get large that is called an ultraviolet (UV) divergence. If a Feynman integral diverges as momenta get small that is called an infrared (IR) divergence. The name comes from light, where large energy is small wavelength and vice versa. We will only talk about UV divergences in this class. Note that there can also be UV subdivergences where only a subset of the momenta getting large causes a problem. We find UV divergences by power counting: how many powers of momentum in the numerator and in the denominator vs how many dk. This is called power counting.

To keep track of this combinatorially we need a few definitions.

Definition 1. *A graph for us is a set of half edges, H along with a set $V(G)$ of disjoint subsets of half edges which partition H and a set $E(G)$ of disjoint pairs of half edges known as* internal edges*. Unpaired half edges are known as* external edges*.*

This is the right way to axiomatize graphs when they are built from Wick's theorem, and is like the def of combinatorial map but forgetting the order information.

Definition 2. *A* combinatorial physical theory *is a set of half-edge types along with*

- *a set of pairs of not necessarily distince half edge types defining the permissible edge types,*
- *a set of multisets of half edge types defining the permissible vertex types,*
- *an integer associated to each edge type and vertex type known as the* power counting weight*, and*
- *$D \geq 0$, the dimension of space time.*

The example we've been working with so far is ϕ^4 where there is one half edge type, one edge type made of a pair of two half edges, and one vertex type made of a multiset of four half edges. The power counting weights are 2 for the edge type and 0 for the vertex type. $D = 4$.

Another important example is quantum electrodynamics (QED). QED has three half edge types, a half-photon, a front-half-fermion and a back-half-fermion. There are two edge types: a pair of half-photons (giving an unoriented edge called the photon), and a pair of one of each half fermion (giving an oriented edge called the fermion). There is one vertex type consisting of one of each half-edge type. The power counting weights are 2 for the photon, 1 for the fermion, and 0 for the vertex. Again $D = 4$.

Definition 3. *A graph G in a combinatorial physical theory is a graph and an edge type for each half edge of the graph so that all edges and vertices are permissible. The* superficial degree of divergence *of G is*

$$sdd(G) = D\ell - \sum_e w(e) - \sum_v w(v)$$

where ℓ is the dimension of the cycle space of G (the "loop number" of G) and the $w(a)$ are the power counting weights.

If $sdd(G) \geq 0$, say that G is divergent*. If $sdd(G) = 0$, say that G is* logarithmically divergent*.*

A logarithmically divergent integral is one that diverges like the integral for the logarithm: $\int dx/x$.

Definition 4. *Given a graph in a theory the multiset of half edge types of the external edges of the graph is the* external leg structure *of the graph.*

Definition 5. *A combinatorial physical theory is* renormalizable *in the combinatorial sense if the superficial degree of divergence of a connected diagram depends only on the external leg structure.*

As with Euler's formula you could instead allow disconnected graphs in the definition but then you'd also get a dependence on the number of components. The usual definition of renormalizable is that you only need a finite number of parameters to renormalize all the graphs; with this definition of renormalizable the point is that you can renormalize all the diagrams with the same external leg structure together which is what happens in renormalizable theories in the physical sense.

These are the notions we'll need for renormalization Hopf algebras.

Next time

Next time we'll define Hopf algebras.

References

Look at section 2.2 of `https://arxiv.org/pdf/0810.2249.pdf`. That's actually my PhD thesis; you can also find the same content with a different final chapter as *Rearranging Dyson-Schwinger equations*, Mem. Amer. Math. Soc. **211**, (2011). Today's material can also be found in chapter 5 of *A combinatorial perspective on quantum field theory*, Springer briefs in mathematical physics, **15**, (2017).