COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 14 SUMMARY

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Summary

Today we derived Feynman rules in ϕ^4. So our Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

and we had

$$Z[J] = \int D\phi \exp(i \int d^4x\mathcal{L} + J\phi)$$

Now what we want to do is try to follow the plan we had in graph counting. Since the path integral isn't well defined we can't do this by rigorous derivations, but rather we will take things by analogy to the finite dimensional case, and view derivations with the path integral as a kind of heuristic.

The first thing is to think about the basic Gaussian integral. Before we had $\int d\nu(A)e^{-\frac{1}{2}x^tAx}$ so what is the analogue of A? To see it we need to actually write out $\int d^4x(\partial\phi)^2$:

$$\begin{aligned}\int d^4x(\partial\phi)^2 &= \int d^4 \sum_{\mu=1}^{4} \partial_{x_\mu}\phi(x)\partial_{x_\mu}\phi(x) \\ &= \sum_{\mu=1}^{4} \int\int d^4xd^4y\partial_{y_\mu}\phi(y)\partial_{x_\mu}\phi(x)\delta(x-y)\end{aligned}$$

where $\delta(x-y)$ is the Dirac delta. Now integrate by parts with $f = \phi(y)$ and $g = \int d^4x\partial_{x_\mu}\phi(x)\delta(x-y)$ as a function of y. Get

$$= -\sum_{\mu=1}^{4} \int\int d^4xd^4y\phi(y)\partial_{y_\mu}\partial_{x_\mu}\delta(x-y)\phi(x) + \text{another term}$$

where we can discount the second term by assuming the fields decay fast enough. Now we're happy because things are written in the form $\phi(\text{operator})\phi$ in analogy with x^tAx. In particular

$$\int d^4x\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}\int\int d^4xd^4y\phi(y)A(x,y)\phi(x)$$

where $A(x,y) = (\sum_{\mu=1}^{4}\partial_{x_\mu}\partial_{y_\mu} + m^2)\delta(x-y)$. This is typically written in the more condensed form

$$\int d^4x\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}\int d^4x\phi(\partial^2 + m^2)\phi$$

In the finite dimensional case the Gaussian integral throws down

$$\int_{\mathbb{R}^N} dx_1 \cdots dx^n e^{i(\frac{1}{2}x^t Ax + Jx)} = \frac{\sqrt{2\pi i}^N}{\sqrt{\det A}} e^{-iJA^{-1}J}2$$

so we still need the analogue of A^{-1}. This will be called the *propagator* and written $D(x-y)$. It depends only on $x-y$, not x and y separately because of translation invariance. Again proceed by analogy with the finite dimensional case. Write $A = (a_{ij})$, $A^{-1} = (d_{ij})$, Then $AA^{-1} = I$ is

$$\sum_j a_{ij} d_{jk} = \delta_{j,k}$$

where this delta is the Kronecker delta. Now take the continuous analogue to get

$$-(\partial^2 + m^2)D(x-y) = \delta(x-y)$$

This is an honest differential equation and can be solved. Get

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2}$$

where a helpful fact to know is $(2\pi)^4 \delta(x) = \int d^4 k e^{ikx}$ where x is a 4-vector and as is usual in this world dot product of vectors is just written as product. Now actually this $D(x-y)$ has a problem at $k^2 = m^2$ and we need to know what to do at the pole. For this reason physicists usually write

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

meaning take the limit of that expression as $\epsilon \to 0$.

Now can proceed as for graph counting and each edge will give a factor of $D(x-y)$. The last thing we need to consider is the vertex. In the graph counting, a vertex coming from a factor of $\lambda q^4/4!$ just contributed a factor of λ. Now the vertex if it is at position v comes from a factor of $i\int d^4 v \lambda \phi^4/4!$. Suppose the four edges coming in are at positions w, x, y, z and the momentum variables (the k variables from the expression for D) are k_1, k_2, k_3, k_3 respectively. Then the vertex and these edges together contribute

$$\begin{aligned} -i\lambda \int d^4 v D(w-v)D(x-v)D(y-v)D(z-v) &= \frac{-i\lambda \int d^4 v e^{i(k_1+k_2+k_3+k_4)v}}{(k_1^2 - m^2)(k_2^2 - m^2)(k_3^2 - m^2)(k_4^2 - m^2)} \\ &= \frac{i\lambda (2\pi)^4 \delta(k_1 + k_2 + k_3 + k_4)}{(k_1^2 - m^2)(k_2^2 - m^2)(k_3^2 - m^2)(k_4^2 - m^2)} \end{aligned}$$

So now we have our Feynman rules. Here is how to write down the Feynman integral for a scalar field theory graph.

Arbitrarily orient the edges of the graph. Each edge e_i gets a variable k_i (the momemtum running through the edge). Each vertex gets a $\delta(\sum \pm k_i)$ where the sum adds the momenta for the edges running in to the vertex and subtracts the ones running out. We integrate $\int \frac{d^4 k_i}{(2\pi)^4}$ for each edge. Equivalently (see assignment) we can pick an edge variable for each element of a basis of the cycle space of the graph, integrate only over those variables, not have any delta functions, and the momentum of an edge is the signed sum of the cycles variables running through it.

Finally what is the integrand? For an edge with momentum p get a factor of $\frac{1}{p^2-m^2}$; for a vertex get a factor $i\lambda$ and take the product of the factors for the edges and the vertices.

Other QFTs are similar but get more complicated (as the factors have tensor indices and are only "factors" in notation because of Einstein summation).

Next time

Next time we'll talk about divergences and power counting.

References

Zee *Quantum field theory in a nutshell* (2003 Princeton).

Peskin and Schroeder *An introduction to quantum field theory* (1995 Westview). The path integral is chapter 9 in Peskin and Schroeder.