COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 12 SUMMARY – PART 1

WINTER 2018

Summary

Today we finished up the proof of the Harer Zagier result and then started the second part of the course. This document is only on the former of those.

From last time we had

Lemma 1. *$\frac{T_n(N)}{(2n-1)!!}$ is a polynomial in n of degree $N-1$*

Now say we colour the vertices of the 2n-gon (with no properness restriction). A gluing *agrees with the colouring* if vertices glued together are of the same colour (but not all vertices of the same colour need to be glued together). Then

$$T_n(N) = \sum_{(\sigma,c)} 1$$

where the sum runs over gluings σ and colourings c in $\leq N$ colours such that σ is compatible with c. The proof is to observe that a colouring that agrees with a gluing is exactly a colouring (again no properness) of the embedded graph of the gluing and there are N^V of these where V is the number of vertices.

Now let $\widetilde{T}_n(N)$ be the same sum as above but with *exactly* N colours. Then we have

$$T_n(N) = \sum_{L=1}^{N} \binom{N}{L} \widetilde{T}_n(L)$$

$$\widetilde{T}_n(N) = \sum_{L=1}^{N} \binom{N}{L} T_n(L)(-1)^{N+L}$$

now observe that $\widetilde{T}_0(N) = \widetilde{T}_1(N) = \cdots = \widetilde{T}_{N-2}(N) = 0$ since there are not enough vertices in the embedded graph for all the colours. Also by plugging in the expression above to the lemma from last time we can write $\widetilde{T}_n(N)$ in terms of $T_n(L)$ and hence obtain that $\widetilde{T}_n(N)$ is also a polynomial in n of degree at most $N-1$. Since we know $N-1$ of its roots we thus know

$$\widetilde{T}_n(N) = C_n n(n-1)(n-2)\cdots(n-N+2)$$

With a little bit of algebra along with the fact that the genus 0 case of $T_n(N)$ is counted by Catalan numbers (because it is given by the non-crossing chord diagrams) we can calculate that $C_{n+1} = 2^n/n!$. Pluging back in to $T_n(N)$ and then summing $T(N,s) = 1 + 2Ns + 2s\sum_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!} s^n$, we get an expression for $T(N,s)$ as a double sum with binomial coefficients. Some manipulations then reduce this to

$$T(N, s) = \left(\frac{1+s}{1-s}\right)^N$$

which is what we wanted to prove.

Next time

This class continued with an introduction to Feynman diagrams. See the second part of the summary.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)

Harer and Zagier *The Euler characteristic of the moduli space of curves* Invent. math. **85**, 457-485 (1986)