COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 11 SUMMARY

WINTER 2018

Summary

Today we did the analytic part of the proof of Harer and Zagier's result. We first reviewed some facts on unitary matrices. We want to diagonalize the Hermitian integral. This takes a bit of work to set up. Consider the map

$$\alpha : \mathbb{R}^N \times U_N \to \mathrm{Herm}(N)$$

given by $\alpha(\Lambda, U) = U^{-1}\Lambda U = H$ where U_N is the group of $N \times N$ unitary matrices. Each H has inverse image $N!$ copies of T where T is the subspace of U_N given by diagonal matrices (which must each be of the form $e^{i\theta}$, $\theta \in \mathbb{R}$). So we can instead consider the map α

$$\mathbb{R}^N \times U_N/T \to \mathrm{Herm}(N)$$

Now the dimensions on both sides match and on the open subsets where the entries of Λ are distinct (so the eigenvalues of H are distinct) it is a covering map. This means that analytically we are set to go with changing variables in the integral in essentially the same way as in multivariate calculus. We just need to compute the Jacobian. A trick to do it nicely is to define dA as on question A4 of assignment 1 and then by differentiating $H = U^{-1}\lambda U$ obtain $dH = U^{-1}LU$ where $L = -\Omega\Lambda + d\Lambda + \lambda\Omega$ and $\Omega = (dU)\cdot U^{-1}$. Now writing $\Omega = (d\omega_{ij})$ and $L = (\ell_{ij}$ we can compute

$$\ell_{ii} = d\lambda_i \quad \text{and} \quad \ell_{ij} = (\lambda_i - \lambda_j)d\omega_{ij} \text{ for } i \neq j$$

where the λ_i are the diagonal entries of Λ. So we get

$$d\nu(H) = d\nu(L) = \prod_{i=1}^{N} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \mathrm{Re}(d\omega_{ij})\mathrm{Im}(d\omega_{ij})$$

so an integral over $\mathrm{Herm}(N)$ of a unitary invariant function can be separated into an integral over $\mathbb{R}^N$ and one over U_n/T but the latter one depends only on N, not on the function you're integrating, so it just contributes to an overall constant.

Now we'll apply this in a particular case

Lemma 1. *$\frac{T_n(N)}{(2n-1)!!}$ is a polynomial in n of degree $N-1$*

The proof runs as follows

$$
\begin{aligned}
T_n(N) &= \int_{\mathrm{Herm}(N)} \mathrm{tr}(H^{2n}) d\mu(H) \\
&= c_N \int_{R^N} \mathrm{tr}(\Lambda^{2n}) \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\mu(\lambda_i) \\
&= c_N \int_{R^N} (\lambda_1^{2n} + \cdots + \lambda_N^{2n}) \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\mu(\lambda_i) \\
&= N c_N \int_{R^N} \lambda_1^{2n} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\mu(\lambda_i)
\end{aligned}
$$

by symmetry. Now expand $\prod_{i<j}(\lambda_i - \lambda_j)^2$ as a polynomial in the λ_i and integrate all the variables other than λ_1. None of these other integrations depend on n so what we get is just a polynomial in λ_1 with coefficients which are constants (for fixed N). Thus $T_n(N)$ is a linear combination of expressions of the form

$$\langle \lambda_1^{2n+2k} \rangle = (2n + 2k - 1)!!$$

Thus $\frac{T_n(N)}{(2n-1)!!}$ is a linear combination of expressions of the form

$$\frac{(2n + 2k - 1)!!}{(2n - 1)!!}$$

which is a polynomial in n of degree k. The largest k which appears is $N - 1$ and the leading coefficient is nonzero which proves the result.

That's the only thing we need the Hermitian integral for. What remains is a counting argument. I made some of the definitions today in class but I'll save it for next class' summary so it is all in the same place.

Next time

Next class we will finish the proof of the Harer Zagier result by counting.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)

Harer and Zagier *The Euler characteristic of the moduli space of curves* Invent. math. **85**, 457-485 (1986)