COMBINATORICS OF FEYNMAN DIAGRAMS LECTURE 10 SUMMARY

WINTER 2018

Summary

Today we finished our quick overview of combinatorial maps. We made the following observations (some without proof)

- A combinatorial map is the same information as a graph embedded in a surface (but don't forget Dehn twists).
- The face permutation is $\phi = \alpha^{-1}\sigma^{-1}$ where the face permutation tells us how the edges permute by taking a consistent half of each edge (in my conventions it was the second half, but my conventions were not awesome, so feel free to use your own as long as you're clear and consistent).
- Euler's formula still works, the numbers needed in it are the numbers of cycles (including fixed points) in the appropriate permutations.
- We can dualize polygon gluings to be gluing the half edges around a star (or corolla, that is, a vertex with half edges). In the Hermitian integral the i_j labelled the vertices of the polygon. In this dual version they label the corners between half-edges in the star.
- We can reinterpret as fat graphs or ribbon graphs. Now each edge has some width, it is a strip, and pairing them is pairing the two sides of the strip. So far the Hermitian constraints mean they are always paired without a twist, but on your homework you are considering a different situation.

By the same sorts of arguments as before we have

Proposition 1.

$$\int_{Herm(N)} (trH)^{\alpha_1}(trH^2)^{\alpha_2}\cdots(tr(H^k)^{\alpha_k}d\mu(H) = \sum_p N^{\textit{number of faces of } p}$$

where the sum runs over pairings p of α_1 1-valent stars, α_2 2-valent stars, ...

$$\int_{Herm(N)} \frac{(trH)^{\alpha_1}(trH^2)^{\alpha_2}\cdots(tr(H^k)^{\alpha_k}}{\alpha_1!\alpha_2!2^{\alpha_2}\cdots\alpha_k!k^{\alpha_k}}d\mu(H) = \sum_M \frac{N^{\textit{number of faces of } M}}{|Aut(M)|}$$

where the sum is over combinatorial maps M with α_i i-valent vertices for $1 \leq i \leq k$.

Note that it really should be i^{α_i} in the denominator for the second equation not $(i!)^{\alpha_i}$ because the second equation derives from the first one and in the first one everything is labelled so we have an order on the half edges at each vertex, but in the second we only want a cyclic order, so we divide by i to forget where the order begins and keep only the cyclic structure.

Finally at the end of class we moved into the last subsection of this first section of the course. Our goal in this last subsection is to prove a closed form for a slightly weirdly normalized generating function for 1-face maps. Specifically let

$$T_n(N) = \sum_{g \geq 0} \epsilon_g(n) N^{n+1-2g}$$

where $\epsilon_g(n)$ is the number of ways to get a surface of genus g from gluing a $2n$-gon, and let

$$T(N,s) = 1 + 2Ns + 2s \sum_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!} s^n$$

then the result is

Theorem 2 (Harer and Zagier)**.**

$$T(N,s) = \left(\frac{1+s}{1-s} \right)^N$$

Next time

Next class we will use the Hermitian integral and some counting to prove that theorem.

References

Lando and Zvonkin "Graphs on surfaces and their applications" (Springer 2004)

Harer and Zagier *The Euler characteristic of the moduli space of curves* Invent. math. **85**, 457-485 (1986)