COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 4

SOLUTIONS

PART A

(1) Following the theorem from class you'd get

$$Q = \left(\frac{X^+}{(X^-)^4}\right)^{2/(4-2)} = \frac{X^+}{(X^-)^4}$$

for ϕ^4 and

$$Q = \left(\frac{X^Y}{(X^-)^3}\right)^{2/(3-2)} = \frac{(X^+)^2}{(X^-)^6}$$

for ϕ^3 where $+$ and Y are representing the vertices based on their shape. Unfortunately if you look closely at the proof or try an example in the way we did QED (or just check for consistency with QED) you'll see that the powers of the edges should involve $n(e)/2$ in place of $n(e)$ as we defined it. Thus the correct answers are

$$Q = \left(\frac{X^+}{(X^-)^{4/2}}\right)^{2/(4-2)} = \frac{X^+}{(X^-)^2}$$

for ϕ^4 and

$$Q = \left(\frac{X^Y}{(X^-)^{3/2}}\right)^{2/(3-2)} = \frac{(X^+)^2}{(X^-)^3}$$

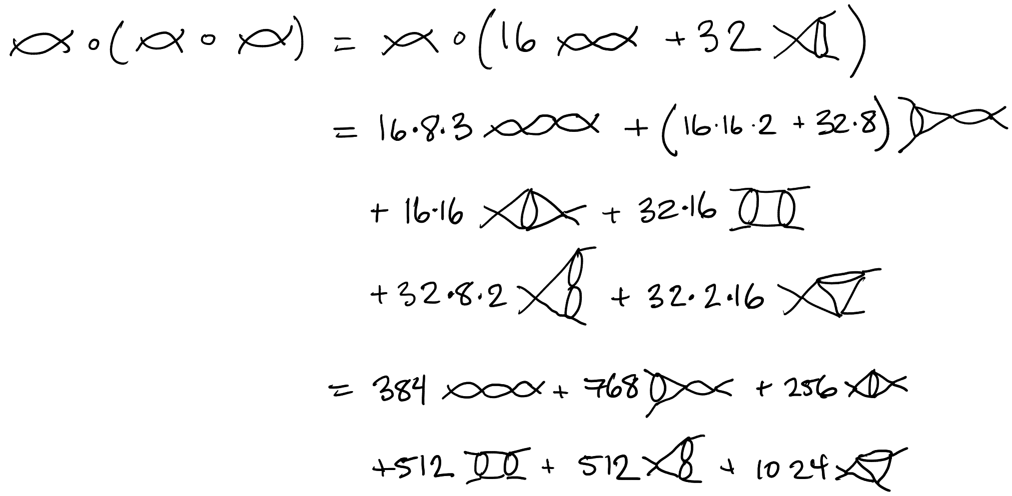
I accepted either answer.

(2) This is a calculation: let's do it

$$\begin{aligned}
&[a,[b,c]] + [b,[c,a]] + [c,[a,b]] \\
&= [a, b\circ c - c\circ b] + [b, c\circ a - a\circ c] + [c, a\circ b - b\circ a] \\
&= a\circ(b\circ c) - a\circ(c\circ b) - (b\circ c)\circ a + (c\circ b)\circ a \\
&\quad + b\circ(c\circ a) - b\circ(a\circ c) - (c\circ a)\circ b + (a\circ c)\circ b \\
&\quad + c\circ(a\circ b) - c\circ(b\circ a) - (a\circ b)\circ c + (b\circ a)\circ c \\
&= a\circ(b\circ c) - (a\circ b)\circ c - a\circ(c\circ b) + (a\circ c)\circ b \\
&\quad + b\circ(c\circ a) - (b\circ c)\circ a - b\circ(a\circ c) + (b\circ a)\circ c \\
&\quad + c\circ(a\circ b) - (c\circ a)\circ b - c\circ(b\circ a) + (c\circ b)\circ a \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

by pre-Lie.

(3) Let's not label the external legs just to keep it simple. Then we can calculate



For those who do want to label the external legs these are evenly distributed among the permuations of the external legs, so it remains to collect those which are isomorphic.

PART B

(1) Let $\mathcal{H}$ be the Connes-Kreimer Hopf algebra over $\mathbb{Q}$. Define $Z : \mathcal{H} \to \mathbb{Q}$ by $Z(f) = \delta_{\bullet,f}$ for any forest f and extended linarly, where δ is the Kronecker delta. Also extend Z to $\mathcal{H}[[x]]$ by acting on coefficients. Prove the following things:

(a) Write $a = \sum a_{i,j} f_{i,j} x^i$ and $b = \sum b_{i,j} f_{i,j} x^i$ where $\{f_{i,j}\}_{j \in J_i}$ runs over all forests with i vertices and these are indexed by the set J_i. Note that there is only one forest with one vertex, namely $\bullet$, so $f_{1,1} = \bullet$ and $J_1 = \{1\}$, and similarly there is only one forest with no vertices, namely 1, so $f_{0,1} = 1$ and $J_0 = \{1\}$. Then calculate

$$Z(ab) = Z((a_{0,1}b_{1,1} + a_{1,1}b_{0,0}) \bullet x + \text{other terms}) = (a_{0,1}b_{1,1} + a_{1,1}b_{0,0})x$$

It works whether you assume Z keeps the x or not. Since I wrote that it extends to the power series by acting on coefficients, I'll keep the x. Now for the other side

$$Z(a)\epsilon(b) + \epsilon(a)Z(b) = a_{1,1}xb_{0,1} + a_{0,1}b_{1,1}x$$

which is the same.

(b) Write $T(x) = \sum_{i \geq 0} t_i x^i$. The DSE is Hopf so $\Delta(t_i) \in A \otimes A$. Use the grading to write $\Delta(t_i) = \sum_{j=0}^{i} t_{i,j}^{(1)} \otimes t_{i,i-j}^{(2)}$ where $t_{i,j}^{(k)}$ has degree j and all the $t_{i,j}^{(k)} \in A$. Then $(Z \otimes \mathrm{Id})(\Delta(t_i)) = Z(t_{i,1}^{(1)}) \otimes t_{i,i-1}^{(2)}$. Next observe that $Z(t_{i,1}^{(1)}) \in \mathbb{Q}$ so use the isomorphism $\mathbb{Q} \otimes A \cong A$ to obtain that $(Z \otimes \mathrm{Id})(\Delta(t_i))$ is a multiple of $t_{i,i-1}^{(2)}$ and hence is in A. This holds for each coefficient of $T(x)$ and hence $(Z \otimes \mathrm{Id}) \circ \Delta(T(x)) \in A[[x]]$ as required.

(c) Write $f(z) = \sum_{i\geq 0} f_i z^i$.
Let's calculate, using part a to keep it from being too ugly.

$$
\begin{aligned}
(Z \otimes \mathrm{Id}) \circ \Delta(T(x)) &= (Z \otimes \mathrm{Id})\Delta(xB_+(f(T(x)))) \\
&= x \sum_{n\geq 0} f_i (Z \otimes \mathrm{Id})\Delta B_+(T(x)^i) \\
&= x \sum_{n\geq 0} f_i Z(B_+(T(x)^i)) + x \sum_{n\geq 0} f_i (Z \otimes B_+)(\Delta(T(x)))^i \\
&= Z(T(x)) + xB_+ \left(\sum_{n\geq 0} f_i (Z \otimes \mathrm{Id})((\Delta(T(x)))^i) \right) \\
&= Z(T(x)) + xB_+ \left(\sum_{n\geq 0} i f_i (\epsilon \otimes \mathrm{Id})((\Delta(T(x)))^{i-1})(Z \otimes \mathrm{Id})\Delta(T(x)) \right) \\
&= Z(T(x)) + xB_+ \left(\sum_{n\geq 0} i f_i T(x)^{i-1} (Z \otimes \mathrm{Id})\Delta(T(x)) \right) \\
&= Z(T(x)) + xB_+(f'(T(x))(Z \otimes \mathrm{Id})\Delta(T(x))) \\
&= Z(T(x)) + L((Z \otimes \mathrm{Id}) \circ \Delta(T(x)))
\end{aligned}
$$

where in the chain of equalities we used the 1-cocycle property and the fact that Z maps to $x\mathbb{Q}$ as well as part a.

(d) Use the previous parts to make this one easy. By part c we have

$$(Z \otimes \mathrm{Id})\Delta(T(x)) = Z(T(x)) + L((Z \otimes \mathrm{Id}) \circ \Delta(T(x))).$$

Rearranging this gives

$$(\mathrm{Id} - L)((Z \otimes \mathrm{Id})\Delta(T(x))) = Z(T(x)).$$

Next note that $\mathrm{Id} - L$ is invertible as a formal power series operation because it begins with 1 and the rest is a multiple of x. So

$$(Z \otimes \mathrm{Id})\Delta(T(x)) = Z(T(x))(\mathrm{Id} - L)^{-1}(1)$$

By part b we know the left hand side is in A and by the grading we know that the left hand side is divisible by x. On the right hand side we have the thing we want to know about multiplied by $Z(T(x))$. As we observed previously $Z(T(x)) \in x\mathbb{Q}$. Furthermore it is nonzero since $f(0) = 1$. Thus dividing both sides by x and a nonzero rational number we obtain that $(\mathrm{Id} - L)^{-1}(1) \in A$.

(2) (a) The equation becomes

$$-\sum_{i\geq 1} i g_i(x) L^{i-1} - \sum_{i\geq 1} \beta(x) \frac{\partial g_i(x)}{\partial x} L^i - \gamma(x)(1 - \sum_{i\geq 1} g_i(x) L^i).$$

Taking the coefficient of L^0 we obtain

$$-g_1(x) - \gamma(x) = 0$$

(and from this you see that my example had a sign error in it coming from different conventions that I was using at the time). More interestingly taking the coefficient of L^i for $i > 0$ we obtain

$$-(i+1)g_{i+1}(x) - \beta(x)\frac{\partial g_i(x)}{\partial x} - \gamma(x)g_i(x) = 0$$

which is the system of differential equations we are looking for.

(b) The base case of the recurrence is $c_1 = 1$ as there is one chord diagram with one chord, then the recurrence is valid for $n \geq 2$. Summing

$$\begin{aligned} C(x) = \sum_{n\geq 1} c_n x^n &= x + \sum_{n\geq 2}\sum_{i=1}^{n-1}(2k-1)c_k c_{n-k} x^n \\ &= x + 2\sum_{n\geq 2}\sum_{i=1}^{n-1} k c_k x^k c_{n-k} x^{n-k} - \sum_{n\geq 2}\sum_{i=1}^{n-1} c_k x^k c_{n-k} x^{n-k} \\ &= x + 2xC'(x)C(x) - C(x)^2 \end{aligned}$$

In the particular case that $\gamma(x) = -g_1(x)$ and $\beta(x) = 2xg_1(x)$ then the equation from part 1 becomes

$$(i+1)g_{i+1}(x) = 2xg_i'(x)g_1(x) - g_i(x)g_1(x)$$

which is similar to the equation for $C(x)$ if we ignore the i's and take all the g_i to be like $C(x)$. This is in fact what happens. The g_i can be viewed as a family of refinements of C and they relate by this generalized form of the classic recurrence. This is discussed in section 4.1 of arXiv:1210.5457.