COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 4

DUE FRIDAY MARCH 16 IN CLASS

PART A

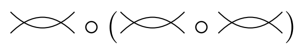
Do any two out of the following three problems for part A.

(1) Give the combinatorial invariant charge (Q) for ϕ^4 theory and ϕ^3 theory (where ϕ^k theory is the scalar field theory with a k-valent vertex).

(2) Suppose that $\circ$ is a pre-Lie product. Define $[a, b] = a \circ b - b \circ a$. Prove that $[,]$ satisfies the Jacobi identity, that is

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

(3) Let $\circ$ be graph insertion as discussed in class. Calculate



PART B

Do all questions from part B.

(1) Let $\mathcal{H}$ be the Connes-Kreimer Hopf algebra over $\mathbb{Q}$. Define $Z : \mathcal{H} \to \mathbb{Q}$ by $Z(f) = \delta_{\bullet,f}$ for any forest f and extended linarly, where δ is the Kronecker delta. Also extend Z to $\mathcal{H}[[x]]$ by acting on coefficients. Prove the following things:

(a) $Z(ab) = Z(a)\epsilon(b) + \epsilon(a)Z(b)$ for all $a, b \in \mathcal{H}[x]$.

(b) Suppose $T(x) = xB_+(f(T(x)))$ is Hopf where is $f(z)$ a formal power series with $f(0) = 1$. Let $A = \mathbb{Q}[t_1, t_2, \ldots]$. Then

$$(Z \otimes \mathrm{Id}) \circ \Delta(T(x)) \in A[[x]].$$

(c) With hypotheses as in the previous part let $L : \mathcal{H}[[x]] \to \mathcal{H}[[x]]$ be defined by $L(a) = xB_+(f'(T(x))a)$. Then

$$(Z \otimes \mathrm{Id}) \circ \Delta(T(x)) = Z(T(x)) + L((Z \otimes \mathrm{Id}) \circ \Delta(T(x)))$$

(d) If $T(x) = xB_+(f(T(x)))$ is Hopf, with notation as in the previous parts, then $(\mathrm{Id} - L)^{-1}(1) \in A[[x]]$.

(2) The *renormalization group equation* explains how change in x and change in L relate for a Green function:

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma(x)\right) G(x, L) = 0$$

For the particular $G(x, L)$ of the original chord diagram expansion, $\gamma = g_1$ and $\beta = 2xg_1$.

(a) Expanding $G(x, L) = 1 - \sum_{i \geq 1} g_i(x)L^i$, rewrite the renormalization group equation as a system of differential equations for the $g_i(x)$.

(b) Let c_n be the number of rooted connected chord diagrams. A classic recurrence for c_n is

$$c_n = \sum_{i=1}^{n-1} (2k-1) c_k c_{n-k}$$

Rewrite this as a differential equation for the generating function $C(x) = \sum c_n x^n$ and compare this differential equation to the differential equations of the previous part.