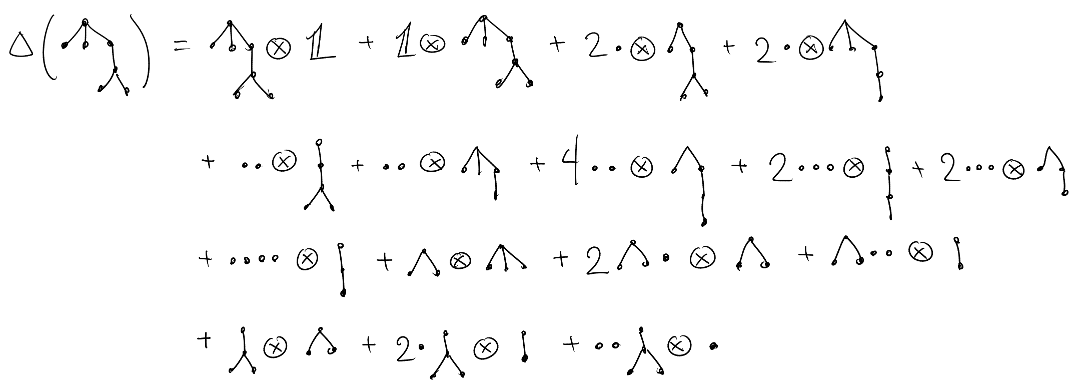
COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 3

SOLUTIONS

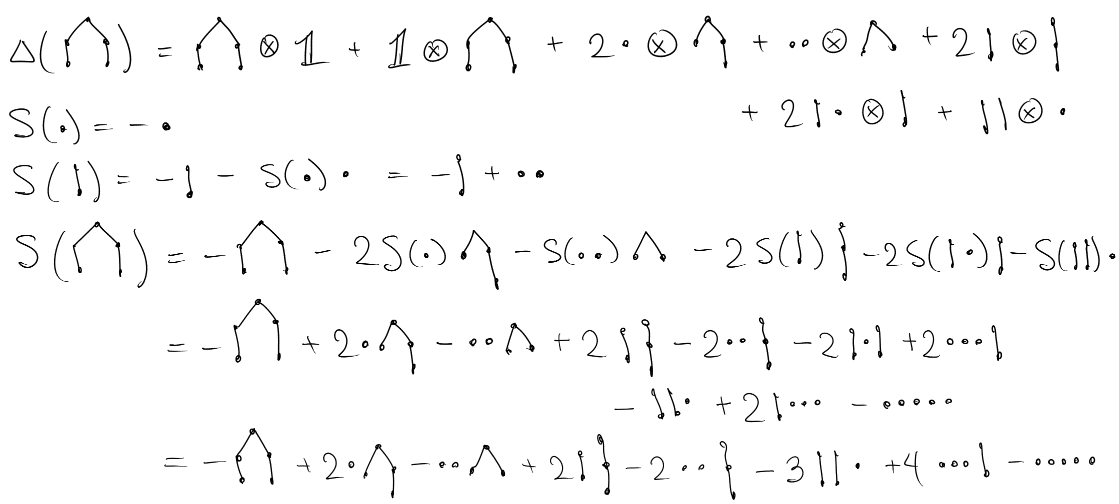
PART A

(1) Ok, I'm a little too lazy to type these ones up:

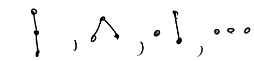
a)



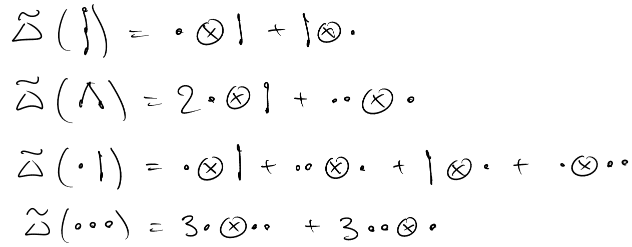
b)



(2)

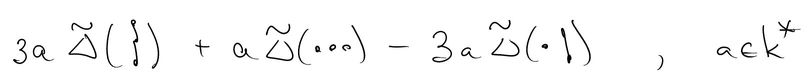
A basis for all elements of order 3 is 

calculate the nonprimitive part of the coproducts of each

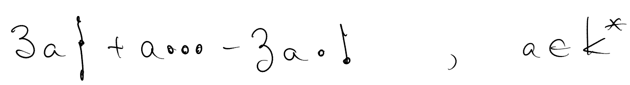


Every linear combination of these equaling 0 gives a primitive element and every primitive element can be built in this way

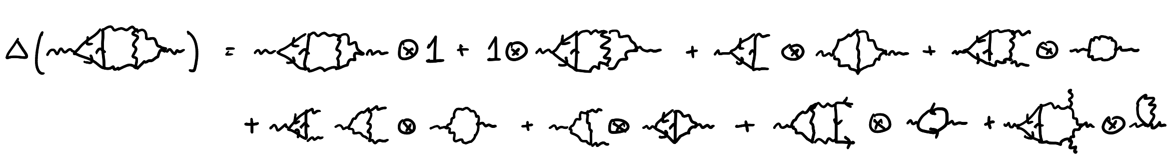
No such linear combination can involve $\tilde{\Delta}(\Lambda)$ since the other three are all cocommutative but it is not. By inspection the only nontrivial linear combinations of the others giving 0 are



So the primitive elements of degree 3 are



(3)



(4) First let's calculate the bits and pieces that we need. Let the whole tree be t. I will label the root by a and the two leaves by b and c. The integration variable for a will always be called z and the integration variables for the leaves will be called z_1 and

z_2 respectively.

$$S_R^{F_s}(b\bullet) = -RF_z(b\bullet) = -\int_0^\infty \frac{dz_1}{s+z_1}$$
$$S_R^{F_s}(c\bullet) = -\int_0^\infty \frac{dz_2}{s+z_2}$$
$$\begin{aligned}S_R^{F_s}(t) &= -RF_s(t) - S_R^{F_s}(b\bullet)RF_s(B_+(c\bullet)) - S_R^{F_s}(c\bullet)RF_s(B_+(b\bullet)) - S_R^{F_s}(c\bullet b\bullet)RF_s(a\bullet)\\ &= -\int_0^\infty\int_0^\infty\int_0^\infty \frac{dzdz_1dz_2}{(1+z)(z+z_1)(z+z_2)} + \int_0^\infty\int_0^\infty\int_0^\infty \frac{dzdz_1dz_2}{(1+z)(1+z_1)(z+z_2)}\\ &\quad + \int_0^\infty\int_0^\infty\int_0^\infty \frac{dzdz_1dz_2}{(1+z)(z+z_1)(1+z_2)} - \int_0^\infty\int_0^\infty\int_0^\infty \frac{dzdz_1dz_2}{(1+z)(1+z_1)(1+z_2)}\end{aligned}$$
$$\begin{aligned}F_{\text{ren}}(t) &= (S_R^{F_s}\star F_s)(t)\\ &= S_R^{F_s}(t) + F_s(t) + S_R^{F_s}(b\bullet)F_s(B_+(c\bullet)) + S_R^{F_s}(c\bullet)F_s(B_+(b\bullet)) + S_R^{F_s}(b\bullet c\bullet)F_s(a\bullet)\\ &= \int_0^\infty\int_0^\infty\int_0^\infty dzdz_1dz_2 \Bigg(-\frac{1}{(1+z_1)(z+z_1)(z+z_2)} + \frac{1}{(1+z)(1+z_1)(z+z_2)}\\ &\quad + \frac{1}{(1+z)(z+z_1)(1+z_2)} - \frac{1}{(1+z)(1+z_1)(1+z_2)} + \frac{1}{(s+z)(z+z_1)(z+z_2)}\\ &\quad - \frac{1}{(s+z)(1+z_1)(z+z_2)} - \frac{1}{(s+z)(z+z_1)(1+z_2)} + \frac{1}{(s+z)(1+z_1)(1+z_2)}\Bigg)\end{aligned}$$

Now let's give all this to Maple

```
> assume(s>0);
> assume(z>0);
> assume(z1>0);
> assume(z2>0);
> integrand := - 1/((1+z)*(z+z1)*(z+z2))\
>  + 1/((1+z)*(1+z1)*(z+z2)) + 1/((1+z)*\
> (z+z1)*(1+z2)) + 1/((s+z)*(z+z1)*(z+z2\
> )) - 1/((s+z)*(1+z1)*(z+z2)) - 1/((s+z\
> )*(z+z1)*(1+z2)) - 1/((1+z)*(1+z1)*(1+z2)) + 1/((s+z)*(1+z1)*(1+z2))\;
                                 1                                 1
integrand := - ------------------------------ + -----------------------------
               (1 + z~) (z~ + z1~) (z~ + z2~)   (1 + z~) (1 + z1~) (z~ + z2~)

                       1                                 1
     + ----------------------------- + ------------------------------
       (1 + z~) (z~ + z1~) (1 + z2~)   (s~ + z~) (z~ + z1~) (z~ + z2~)

                       1                                 1
     - ------------------------------ - ------------------------------
       (s~ + z~) (1 + z1~) (z~ + z2~)   (s~ + z~) (z~ + z1~) (1 + z2~)

                      1                                1
```

```
     - ---------------------------- + ----------------------------
       (1 + z~) (1 + z1~) (1 + z2~)   (s~ + z~) (1 + z1~) (1 + z2~)

> factor(normal(integrand));
                                       2
                              (z~ - 1)  (s~ - 1)
           - ------------------------------------------------------------
             (1 + z~) (z~ + z1~) (z~ + z2~) (1 + z1~) (1 + z2~) (s~ + z~)
> int(factor(normal(integrand)), z2=0..infinity);
                                   2                2      2   2         3
- ln(z~) (s~ z~ - s~ - z~ + 1)/(s~ z1~  z~ + s~ z1~ z~  + z1~  z~  + z1~ z~

             2                      2      2               2     3
     + s~ z1~  + 2 s~ z1~ z~ + s~ z~  + z1~  z~ + 2 z1~ z~  + z~  + s~ z1~

                         2
     + s~ z~ + z1~ z~ + z~ )

> factor(int(factor(normal(integrand)), z2=0..infinity));
                          ln(z~) (z~ - 1) (s~ - 1)
                   - ----------------------------------------
                     (z~ + z1~) (1 + z~) (1 + z1~) (s~ + z~)

> int(factor(int(factor(normal(integrand)), z2=0..infinity)), z1=0..infinity);
                                   2
                              ln(z~)  (s~ - 1)
                          - ----------------------
                                     2
                             s~ z~ + z~  + s~ + z~

> factor(int(factor(int(factor(normal(in\
> tegrand)), z2=0..infinity)), z1=0..infinity));
                                   2
                              ln(z~)  (s~ - 1)
                           - ------------------
                             (1 + z~) (s~ + z~)

> int(factor(int(factor(int(factor(norma\
> l(integrand)), z2=0..infinity)), z1=0..infinity)), z=0..infinity);
                                   2      2
                        -1/3 (ln(s~)  + Pi ) ln(s~)
```

PART B

(1) (a) Let's just recall what it means for a map $f : B \to A$ to be a K-algebra homomorphism: we need

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ m_B \uparrow & & m_A \uparrow \\ B \otimes B & \xrightarrow{f \otimes f} & A \otimes A \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{\text{id}} & K \\ u_B \downarrow & & u_A \downarrow \\ B & \xrightarrow{f} & A \end{array}$$

commuting.

In the case of this question $B = A \otimes A$ and $f = m_A$. The second diagram holds automatically by the unit property in A, so the question is actually asking us to show that the first diagram with $B = A \otimes A$ and $f = m_A$ is equivalent to A being commutative.

Let τ be the transposition map ($\tau(a \otimes b) = b \otimes a$). It's important to note that the multiplication on $A \otimes A$ is not $m_A \otimes m_A$, that doesn't satisfy the correct properties, rather it is $(m_A \otimes m_A) \circ (\text{id} \otimes \tau \otimes \text{id})$.

Suppose the first diagram commutes, then in particular it commutes when applied to $1 \otimes a \otimes b \otimes 1$ for any $a, b \in A$. In that case tracing the diagram we get $m_A(a \otimes b) = m_A(b \otimes a)$ which is the definition of A being commutative. Suppose A is commutative, note that multiplication is also associative and so

$$\begin{aligned} m_A \circ (m_A \otimes m_A) &= m_A \circ (\text{id} \otimes m_A) \circ (\text{id} \otimes m_A \otimes \text{id}) \\ &= m_A(\text{id} \otimes m_A) \circ (\text{id} \otimes (m_A \circ \tau) \otimes \text{id}) \\ &= m_A(\text{id} \otimes m_A) \circ (\text{id} \otimes m_A \otimes \text{id}) \circ \circ (\text{id} \otimes \tau \otimes \text{id}) \\ &= m_A \circ (m_A \otimes m_A) \circ (\text{id} \otimes \tau \otimes \text{id}) \\ &= m_A \circ m_{A \otimes A} \end{aligned}$$

which is the commutativity of the first diagram.

(b) The counit property says $m(\epsilon \otimes \text{id})\Delta = \text{id}$ where the m there is scalar multiplication. Applying this to x we get

$$x = \epsilon(1)x + \epsilon(x)1 = x + \epsilon(x)$$

where $\epsilon(1) = 1$ because we are in a bialgebra. Thus $\epsilon(x) = 0$.

(c) For x primitive by the previous part

$$\begin{aligned} 0 = u\epsilon(x) &= m(\text{id} \otimes S)\Delta(x) \\ &= xS(1) + 1S(x) = x + S(x) \end{aligned}$$

Thus $S(x) = -x$.

(2) Answers will vary, but the key is the use of induction on trees. Here is one possible answer.

Showing the map exists is done inductively on the trees using B_+. Showing the map has the appropriate algebraic properties takes more effort, but the same idea works; since the map is at least an algebra homomorphism then we have enough information for the induction (that is we can use linearity to get ourselves down to forest, then use multiplicativity to get down to trees, then use B_+ returning to forests, all of that only needed algebra properties). This is then done for each additional property that needs proving.