COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 2

SOLUTIONS

PART A

(1) Ok, the 2π and $i\lambda$ factors were all over the place, but I wasn't very careful about them either, so don't worry about it. I'll write it after processing the delta functions because that's usually nicer. Take a basis of the cycle space to be the outer square running clockwise k_1, the lower left triangle running clockwise k_2 and the lower right triangle running clockwise k_3 and send all the external momenta along the sides and the bottom. Then we get

$$\frac{\lambda^4}{(2\pi)^8}\int d^4k_1 d^4k_2 d^4k_3 \frac{1}{(k_1^2 - m^3 + i\epsilon)((q_1+q_2-k_1-k_2-k_3)^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)}$$
$$\cdot\frac{1}{((q_1-k_1-k_2)^2 - m^3 + i\epsilon)((q_4-k_1-k_3) - m^2 + i\epsilon)(k_3^2 - m^2 + i\epsilon)}$$

with the constraint that $q_1 + q_2 = q_3 + q_4$.

(2) No one did this question at all. I think maybe I didn't phrase it in a nice clean way and so that was discouraging. Here's one way to phrase an answer.

Let the graph be G. Let $C_1, C_2, \ldots, C_\ell$ be a basis of the cycle space with the C_i oriented cycles. Let k_i be a variable for C_i. For each cycle write $k_i = \sum_{e\in C_i} \pm p_e$ where the sign is $+$ when the edge is oriented along the cycle and $-$ when it is oriented in the opposite direction. This is a system of equations which we can write in matrix form $\overline{k} = C\overline{p}$ where C is the coefficient matrix and $\overline{k}$ and $\overline{p}$ are the vectors of the k_i and p_i respectively. Now we also have another system of equations, the one given by the vertex identities. Write this one as $\overline{0} = V\overline{p}$.

Now what is it that we want to show in terms of these matrices? We want that the relations between the p_i are compatible with the k_i being free, that is we want $CV^t = 0$. This is true because every cycle comes in and out of every vertex exactly once. Note that another way to interpret the equation $CV^t = 0$ is that the row spaces of C and V are orthogonal. What's left to show is that the row spaces of C and V together span $\mathbb{R}^{|E(G)|}$, so the k_i are exactly the free variables, not less. Note that by definition C has rank the dimension of the cycle space of the graph. Now consider V. Notice that V is the signed incidence matrix of the graph. In particular it has rank $|V(G)| - 1$. To see this note that V has $|V(G)|$ rows and that the sum of the rows is 0, so the rank is at most $|V(G)| - 1$; now since G is connected take a spanning tree of T; the columns of V corresponding to this tree are independent (inductively from the leaves) and so the rank of V is exactly $|V(G)| - 1$. Then by Euler's formula the ranks of V and C sum to $|E(G)|$ and so together span $\mathbb{R}^{|E(G)|}$.

(3) Let G be a graph in this theory with e internal edges, q external edges, and loop number ℓ. For the theory to be renormalizable we need $D\ell - 2e$ to depend only on the number of external edges. By Euler's formula along with regularity we have

$$e(2-k) + k\ell = k - q$$

so for the theory to be renormalizable we need $D = 2k/(k-2)$. Note that D will be an integer only for $k = 3, 4, 6$. Our usual ϕ^4 theory is one of those values (and specifically the one for $D = 4$).

PART B

(1) (a) The proof I had in mind was the inductive one (on edges). You have to take a bit of care with bridges and people did it in different ways. Removing any edge, bridge or not, leaves $\sum w(e) + \sum w(v)$ unchanged, but then either you need to justify why this does what we want in the brige case or you need to ignore bridges for now and have a more complicated base case.

What ended up being more beautiful is what one person did and just computes it directly something like this: Let G be a connected QCD graph with g internal gluon edges, f internal fermion edges, h internal ghost edges, t 3-gluon vertices, q 4-gluon vertices, and r fermion gluon vertices and s ghost gluon vertices. Then

$$\mathrm{sdd}(G) = 4\ell - 2g - f - h + t$$

Now using Euler's formula to rewrite ℓ we have

$$\begin{aligned}\mathrm{sdd}(G) &= 4(g + f + h - t - q - r - s + 1) - 2g - f - h + t \\ &= 2g + 3f + 3h - 3t - 4q - 4r - 4s + 4 \\ &= (2g - 3t - 4q - r - s) + (3f - 3r) + (3h - 3s) + 4\end{aligned}$$

The number of external gluon edges is the total number of gluon half edges minus the number of half edges paired into internal edges. So let e_g be the number of external gluon edges and we have

$$e_g = 4q + 3t + r + s - 2g$$

Similarly if e_f is the number of external fermion edges and e_h the number of external ghost edges then

$$e_f = 2r - 2f \qquad e_h = 2s - 2h$$

Thus

$$\mathrm{sdd}(G) = -e_g - \frac{3}{2}e_f - \frac{3}{2}e_h + 4$$

and so QCD is renormalizable in the combinatorial sense in $D = 4$.

(b) However you did part a you now know that adding external edges decreases the superficial degree of divergence, so there will only be finitely many divergent external structures. With the approach to part a which is above, we can just find the values of e_g, e_g and e_h which work, observing that e_f and e_h must both be even. The result is that the divergent external leg structures in QCD are: vacuum graphs (no external edges), one external gluon, two external gluons, three external gluons, four external gluons, one incoming and one outgoing external

fermion, one incoming and one outgoing external ghost (except actually you never have external ghosts, but physics tells you that, not the combinatorics), one gluon along with one incoming and one outgoing external fermion, and one gluon along with one incoming and one outgoing external ghost.
Note that the divergent structures are the edges and vertices of the theory along with the vacuum graphs and the single gluon.

(2) (a) You just need an example here. How about a 3-graviton vertex (superficial degree of divergence 0) and a 5-graviton vertex with two of the half edges joined into a graph theorist's loop (superficial degree of divergence $4-2=2$). These have the same external leg structure but different superficial degree of divergence.

(b) The vertices of each degree will do the job, or if you prefer, how about a graph-theorists' loop with any number of external edges hanging off the vertex. The superficial degrees of divergence are 0 for the plain vertices and 2 for the vertices with a loop. Divergent in either case.

(c) We can do the same sort of thing as the previous question. Let G be a pure gravity graph with e internal edges and v vertices. Then

$$\mathrm{sdd}(G) = 4\ell - 2e + 2v = 2\ell + 2(e - v + 1) - 2e + 2v = 2\ell + 2$$

by Euler's formula, where ℓ is the dimension of the cycle space of G.

(d) There's not actually much to say here because from the previous part all graphs are divergent, so in particular all 2-edge-connected graphs are. Thus the only way to have a 2-edge-connected graph with no divergent proper 2-edge-connected subgraphs is if it is just a single cycle potentially with external edges.
The point of this question is that such graphs will be primitive in the renormalization Hopf algebra.