COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 1

SOLUTIONS

PART A

(1) (a)

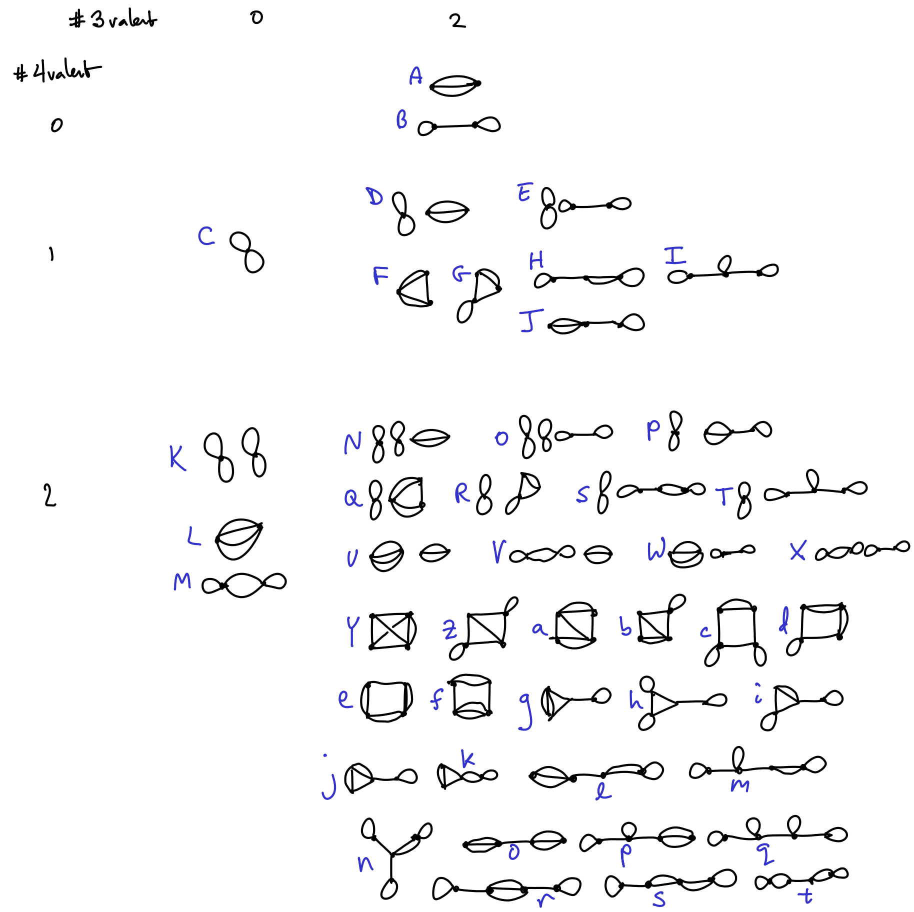
$$\left(\frac{d}{dx}A(x)\right)B(x)+A(x)\left(\frac{d}{dx}B(x)\right)$$
$$=\left(\sum_{n=1}^{\infty} na_nx^{n-1}\right)\left(\sum_{n=0}^{\infty} b_nx^n\right)+\left(\sum_{n=0}^{\infty} a_nx^n\right)\left(\sum_{n=1}^{\infty} nb_nx^{n-1}\right)$$
$$=\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right)\left(\sum_{n=0}^{\infty} b_nx^n\right)+\left(\sum_{n=0}^{\infty} a_nx^n\right)\left(\sum_{n=0}^{\infty}(n+1)b_{n+1}x^n\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)a_{k+1}b_{n-k}\right)x^n+\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}(n-\ell+1)a_kb_{n-\ell+1}\right)x^n$$
$$=\sum_{n=0}^{\infty}\left(\sum_{\ell=1}^{n+1}\ell a_\ell b_{n-\ell+1}\right)x^n+\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}(n-\ell+1)a_kb_{n-\ell+1}\right)x^n$$
$$=\sum_{n=0}^{\infty}\left(\sum_{\ell=1}^{n}\ell a_\ell b_{n-\ell+1}+(n+1)a_{n+1}b_0+\sum_{\ell=1}^{n}(n-\ell+1)a_kb_{n-\ell+1}+(n+1)a_0b_{n+1}\right)x^n$$
$$=\sum_{n=0}^{\infty}\left(\sum_{\ell=1}^{n}(\ell+n-\ell+1)a_\ell b_{n-\ell+1}+(n+1)a_{n+1}b_0+(n+1)a_0b_{n+1}\right)x^n$$
$$=\sum_{n=0}^{\infty}(n+1)\left(\sum_{\ell=0}^{n+1}a_\ell b_{n+1-\ell}\right)x^n$$
$$=\frac{d}{dx}\left(A(x)B(x)\right)$$

(you don't need to include that many steps unless you want to)

(b)

$$\frac{d}{dx}\exp(x)=\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=1}^{\infty}\frac{n}{n!}x^{n-1}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$
$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\exp(x)$$

(2) We have zero, one, or two 3-valent vertices and zero, one, or two 4-valent vertices. First let's just draw all the unlabelled graphs and then think about the coefficients in each case. I'm too lazy to type draw these properly, so here are some hand drawings.



Now let's think about the coefficients and actually write out the expansion.

$$1 + \left(\frac{1}{12}A + \frac{1}{8}B\right)\lambda_1^2 + \frac{1}{8}C\lambda_2$$

$$+ \left(\frac{1}{96}D + \frac{1}{64}E + \frac{1}{8}F + \frac{1}{8}G + \frac{1}{8}H + \frac{1}{16}I + \frac{1}{12}J\right)\lambda_1^2\lambda_2$$

$$+ \left(\frac{1}{128}K + \frac{1}{48}L + \frac{1}{16}M + \frac{1}{128}N\right)\lambda_2^2$$

$$+ \left(\frac{1}{1536}N + \frac{1}{1024}O + \frac{1}{96}P + \frac{1}{64}Q + \frac{1}{64}R + \frac{1}{64}S + \frac{1}{128}T\right.$$

$$+\frac{1}{552}U+\frac{1}{192}V+\frac{1}{384}W+\frac{1}{128}X$$
$$+\frac{1}{8}Y+\frac{1}{16}Z+\frac{1}{8}a+\frac{1}{4}b+\frac{1}{16}c+\frac{1}{16}d+\frac{1}{16}e+\frac{1}{24}f$$
$$+\frac{1}{24}g+\frac{1}{16}h+\frac{1}{8}i+\frac{1}{8}j+\frac{1}{32}k$$
$$+\frac{1}{24}\ell+\frac{1}{16}m+\frac{1}{32}n+\frac{1}{72}o+\frac{1}{24}p+\frac{1}{32}q+\frac{1}{48}r+\frac{1}{16}s+\frac{1}{32}t\Big)\lambda_1^2\lambda_2^2+\cdots$$

Ok that was a bit annoying, but hopefully you learned something.

(3) $\mathsf{Seq}(\mathcal{Z})$ means all labelled sequences of atoms. A sequence is an ordered k-tuples for some $k \geq 0$ of atoms. Each k-tuple contains k atoms and a labelling is a bijection between these atoms and $\{1, 2, \dots, k\}$. So what we really have is all orderings of $\{1, 2, \dots, k\}$ for all $k \geq 0$. This is exactly permutations in one line notation. From the constructions we discussed in class we know that the exponential generating function is $\frac{1}{1-Z(x)} = \frac{1}{1-x}$.

$\mathsf{Set}(\mathsf{Cyc}(\mathcal{Z}))$ means all labelled sets of cycles of atoms. A labelled cycle of atoms is a cyclic order on k atoms for some $k \geq 0$ along with a bijection between these atoms and $\{1, 2, \dots, k\}$. When we form a set of these with the labelled set construction we must relabel each cycle so as to be compatible with the original cyclic order but so that the labels on all the cycles of the set are distinct and consecutive. What this means is that an element of $\mathsf{Set}(\mathsf{Cyc}(\mathcal{Z}))$ of size n is a set partition of $\{1, 2, \dots, n\}$ where each part has a cyclic structure. This is exactly permutations encoded by their cycle structure. Thus both describe permutations. Finally, from the constructions we discussed in class we know that in this second case the exponential generating function is $\exp(\log(\frac{1}{1-x})) = \frac{1}{1-x}$ so they agree as they should.

(4)
- Write $A = (a_{ij})$ and $B = (b_{ij})$. Then

$$d(AB) = (d\sum_j a_{ij}b_{jk}) = (\sum_j((da_{ij})b_{jk}+a_{ij}(db_{jk}))) = (\sum_j(da_{ij})b_{jk})+(\sum_j a_{ij}(db_{jk})) = (dA)B+A(dB)$$

- $0 = d(AA^{-1}) = (dA)A^{-1} + A(dA^{-1})$ so $d(A^{-1}) = -(A^{-1})(dA)A^{-1}$.

PART B

(1) (a) Let $A, B \in S_N$. For any real numbers a and b, $(aA+bB)^t = aA^t+bB^t = aA+bB$ and so $aA+bB$ is also symmetric. For $A = (a_{ij})$ we have $a_{ij} = a_{ji}$ and no other conditions, so we can take as free variables the a_{ij} with $i \leq j$ giving a dimension of $(N^2+N)/2$. More formally you could explicitly define a basis of E_i for $1 \leq i \leq N$ and $F_{i,j}$ for $1 \leq i < j \leq N$ with E_i being the $N \times N$ matrix which is all zero except for a 1 in position i, i and $F_{i,j}$ is the $N \times N$ matrix which is all zero except for a 1 position in i, j and a 1 in position j, i. These are clearly linearly independent since their nonzero entries are disjoint and they span since we can write $A = (a_{ij}) \in S_N$ as $A = \sum_{i=1}^N a_{ii}E_i + \sum_{1\leq i<j\leq N} a_{ij}F_{i,j}$.

(b) First for $S \in S_N$, $\mathrm{tr}(SS^t) = \mathrm{tr}(S^2)$ so this is the usual trace inner product. Now write it out explicitly. Let $S = (s_{ij}) \in S_N$, then

$$\mathrm{tr}(S^2) = \sum_{i=1}^{N}\sum_{j=1}^{N} s_{ij}s_{ji} = \sum_{i=1}^{N} s_{ii}^2 + 2\sum_{1\le i<j\le N} s_{ij}^2$$

This is manifestly a quadratic form as it contains only quadratic terms in the variables. It is nondegenerate because it consists of a nonzero number of squares of real numbers.

(c) We can just read the matrix off of the expression for the quadratic form in the previous question. The part that can trip you up the first time is that now this new matrix is $(N^2+N)/2 \times (N^2+N)/2$ and is indexed by the s_{ij}. So put the s_{ii} first and then the s_{ij} for $i<j$ and obtain the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 2 \end{bmatrix}$$

Thus the covariance matrix is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix}$$

and so $\langle s_{ii}\rangle = 1$ for $1 \le i \le N$ and $\langle s_{ij}\rangle = \frac{1}{2}$ for $1 \le i < j \le N$.

(d) Given $\langle s_{ij}s_{kl}\rangle$ by swapping the indices of both if necessary we can assume that $i \le j$. If $k \le l$ then $\langle s_{ij}s_{kl}\rangle$ is directly given by the entry of the covariance matrix indexed by (ij, kl) which is zero unless $i = k$ and $j = l$ since the covariance matrix has no off-diagonal entries. If $k > l$ then $\langle s_{ij}s_{kl}\rangle = \langle s_{ij}s_{lk}\rangle$ which is the entry of the covariance matrix indexed by (ij, lk). Similarly this is zero unless $i = l$ and $j = k$.

(e) We know the normalizing constant for a $(N^2+N)/2$-dimensional Gaussian integral is

$$\frac{\sqrt{\det(B)}}{\sqrt{2\pi}^{\frac{N^2+N}{2}}}$$

and in this case

$$\det(B) = \left(\frac{1}{2}\right)^{\frac{N^2-N}{2}}$$

so the constant is

$$\frac{\sqrt{\left(\frac{1}{2}\right)^{\frac{N^2-N}{2}}}}{\sqrt{2\pi}^{\frac{N^2+N}{2}}} = \frac{1}{(\sqrt{\pi})^{\frac{N^2+N}{2}}\sqrt{2}^N}$$

(f) Here we go

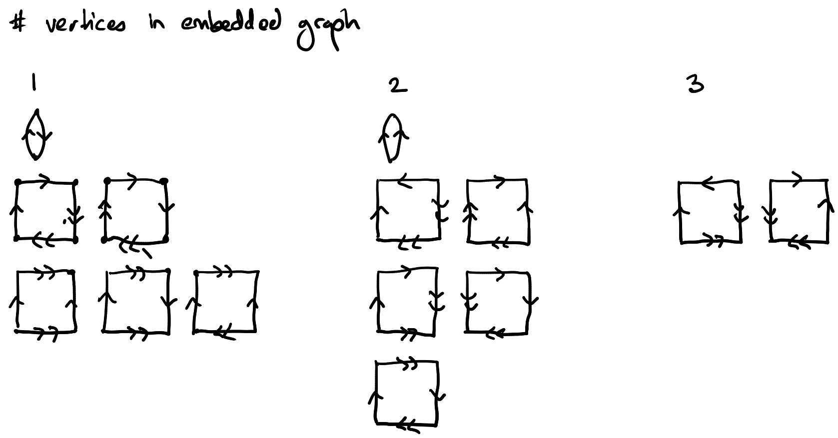
$$\begin{aligned}
\int_{S_N} \operatorname{tr}(S^2)d\mu(S) &= \int_{S_N} \sum_{1\le i,j\le N} s_{ij}s_{ji}d\mu(S) \\
&= \sum_{1\le i,j\le N} \langle s_{ij}s_{ji}\rangle \\
&= \sum_{1\le i\le N} \langle s_{ii}^2\rangle + 2\sum_{1\le i<j\le N}\langle s_{ij}^2\rangle \\
&= N + \frac{2}{2}\frac{N(N-1)}{2} \\
&= \frac{N^2+N}{2}
\end{aligned}$$

The next one is more trouble to get all the 2s right, as you noticed.

$$\begin{aligned}
\int_{S_N} \operatorname{tr}(S^4)d\mu(S) &= \int_{S_N} \sum_{1\le i,j,k,l\le N} s_{ij}s_{jk}s_{kl}s_{li}d\mu(S) \\
&= \sum_{1\le i,j,k,l\le N} \langle s_{ij}s_{jk}s_{kl}s_{li}\rangle \\
&= \sum_{1\le i,j,k,l\le N} \left(\langle s_{ij}s_{jk}\rangle\langle s_{kl}s_{li}\rangle + \langle s_{ij}s_{kl}\rangle\langle s_{jk}s_{li}\rangle + \langle s_{ij}s_{li}\rangle\langle s_{jk}s_{kl}\rangle\right) \\
&= \Bigg(\sum_{1\le i=j=k=l\le N} \langle s_{ij}s_{jk}\rangle\langle s_{kl}s_{li}\rangle + \sum_{1\le i=j=k\ne l\le N} \langle s_{ij}s_{jk}\rangle\langle s_{kl}s_{li}\rangle \\
&\qquad + \sum_{1\le i=k=l\ne j\le N} \langle s_{ij}s_{jk}\rangle\langle s_{kl}s_{li}\rangle + \sum_{1\le i=k, i\ne j, i\ne l\le N} \langle s_{ij}s_{jk}\rangle\langle s_{kl}s_{li}\rangle\Bigg) \\
&\quad + \Bigg(\sum_{1\le i=j=k=l\le N} \langle s_{ij}s_{kl}\rangle\langle s_{jk}s_{li}\rangle + \sum_{1\le i=k\ne j=l\le N} \langle s_{ij}s_{kl}\rangle\langle s_{jk}s_{li}\rangle\Bigg) \\
&\quad + \Bigg(\sum_{1\le i=j=k=l\le N} \langle s_{ij}s_{li}\rangle\langle s_{jk}s_{kl}\rangle + \sum_{1\le j=l=i\ne k\le N} \langle s_{ij}s_{li}\rangle\langle s_{jk}s_{kl}\rangle \\
&\qquad + \sum_{1\le j=l=k\ne i\le N} \langle s_{ij}s_{li}\rangle\langle s_{jk}s_{kl}\rangle + \sum_{1\le j=l, j\ne i, j\ne k\le N} \langle s_{ij}s_{li}\rangle\langle s_{jk}s_{kl}\rangle\Bigg) \\
&= \left(N + \frac{N(N-1)}{2} + \frac{N(N-1)}{2} + \frac{N(N-1)^2}{4}\right) + \left(N + \frac{N(N-1)}{4}\right)
\end{aligned}$$

$$+\left(N+\frac{N(N-1)}{2}+\frac{N(N-1)}{2}+\frac{N(N-1)^2}{4}\right)$$
$$=\frac{2N^3+5N^2+5N}{4}$$

(g)



(h) The gluings are all the chord diagrams where for each chord we can glue one of two ways, so there are $2^n(2n-1)!!$ of them.

(i) This proof is essentially the same as the one we did in class for the Hermitian/orientable case.

$$\int_{S_N}\mathrm{tr}(S^{2n})d\mu(S)=\int_{S_N}\sum_{1\le i_1\ldots i_{2n}\le N}s_{i_1i_2}\cdots s_{i_{2n-1}i_{2n}}s_{i_{2n}i_1}d\mu(S)$$
$$=\sum_{1\le i_1\ldots i_{2n}\le N}\langle s_{i_1i_2}\cdots s_{i_{2n-1}i_{2n}}s_{i_{2n}i_1}\rangle$$

Now use Wick's theorem to rewrite the expression above as a sum over pairings. For each pair in a given pairing there are two ways it can be nonzero, specifically if the pair is $\langle s_{i_ji_{j+1}}s_{i_ki_{k+1}}\rangle$ then either $i_j=i_k$ and $i_{j+1}=i_{k+1}$ or $i_j=i_{k+1}$ and $i_{j+1}=i_k$. Note that if $i_j=i_{j+1}$ then these two cases are the same (and so getting twice the coefficient from the covariance matrix makes sense and we can divide the equality cases up into the two parts and so not have to worry about separately counting nonequal cases from the equalities that they include). Similarly consider a $2n$-gon and label the corners by $i_1, i_2, \ldots, i_{2n}$. Take a gluing of the $2n$-gon where each pairing can be glued in either of the two ways. The equations between the i_j are the same in either view of the situation and the number of free variables is the number of vertices of the embedded graph or equivalently the power of N, likewise to what we did in class. This gives the sum over $2n$-gon gluings both orientable and nonorientable that we want. We can go to stars by dualizing (but see the next question for how to keep the orientation information.

(2) The situtation is the same as for the combinatorial maps except that we need to encode the two possibilities for the orientation of each pair in the gluing. We can do this with ribbon or fat graphs where each half edge now has two sides and two half edges can be glued together with sides matching in either of the two possible ways. This ends up giving graphs which are fattened and edges may or may not have one half twist in them. The one-face maps specifically would still have one face (you can determine the faces directly from the graph just by tracing the sides of the edges until you return to where you began). Or dually, you could take one-vertex ribbon graphs.