COMBINATORICS OF FEYNMAN DIAGRAMS, WINTER 2018, ASSIGNMENT 1

DUE FRIDAY JAN 26 IN CLASS

PART A

Do any three of the four problems from part A.

These problems either concern background things or are routine exercises (possibly both). They will be better exercises for you if you do the ones on the things you *don't* already know, but I leave that choice up to you.

(1) (a) Prove that the product rule of calculus holds for formal power series. This should be a purely formal power series proof. Define any operations or tools for formal power series that you end up needing that were not defined in class.

(b) Prove that for formal power series $\frac{d}{dx}\exp(x) = \exp(x)$ Again, this should be a purely formal power series proof and you should define anything you end up needing that was not defined in class.

(2) Write out the first terms of

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq e^{(-\frac{1}{2}q^2+\frac{\lambda_1 q^3}{3!}+\frac{\lambda_2 q^4}{4!})}$$

as a sum over graphs; specifically give the terms for $\lambda_1^i\lambda_2^j$ with $0 \le i \le 2$, $0 \le j \le 2$.

(3) Here are two specifications for permutations as a labelled combinatorial class. Explain what the each mean and why they are both true:

- $\mathcal{P} = \mathsf{Seq}(\mathcal{Z})$
- $\mathcal{P} = \mathsf{Set}(\mathsf{Cyc}(\mathcal{Z}))$

Read off the equations for the exponential generating function of $\mathcal{P}$ in each case (observe that they are the same).

(4) (This is exercise 3.2 16 from Lando and Zvonkin *Graphs on Surfaces and their applications*). For an $N \times N$ matrix $A = (a_{i,j})$ let dA be the matrix $(da_{i,j})$.

- Prove the product rule for this differential: for any two $N \times N$ matrices A and B, $d(AB) = (dA)B + A(dB)$.
- Prove that for any invertible $N \times N$ matrix A, $d(A^{-1}) = -A^{-1} \cdot dA \cdot A^{-1}$.

PART B

Do all questions from part B.

(1) (Exercises 3.2.9 and 3.2.12 from Lando and Zvonkin):

From "Graphs on surfaces and their applications" by Lando and Zvonkin

Exercise 3.2.9. Here we start a series of exercises which allow the reader to develop on his/her own the theory of integration over the space of symmetric matrices, which will be parallel to that of Hermitian matrices.

Let $\mathcal{S}_N$ denote the space of real symmetric $N \times N$ matrices, i.e., for a matrix $S = (s_{ij}) \in \mathcal{S}_N$, we have $s_{ij} = s_{ji}$ for all $i, j = 1, \dots, N$.

1. Verify that $\mathcal{S}_N$ is a vector subspace in the space of all real $N \times N$ matrices. Find the dimension of this subspace.

2. Verify that the trace form $\mathrm{tr}(S^2)$ determines a non-degenerate quadratic form on the space of symmetric matrices coinciding with the restriction of the ordinary scalar product $(X, Y) = \mathrm{tr}(XY^t)$ in the space of all real matrices.

3. Find the matrix of the trace form $\mathrm{tr}(S^2)$ and the scalar products $\langle s_{ij}^2 \rangle = \langle s_{ij} s_{ji} \rangle$.

4. Prove that $\langle s_{ij} s_{kl} \rangle \neq 0$ if and only if either $i = k$, $j = l$ or $i = l$, $j = k$.

In what follows we denote by

$$d\mu(S) = c_N \exp\{-\frac{1}{2}\,\mathrm{tr}(S^2)\} dv(S)$$

the Gaussian measure on the space of symmetric matrices. Here

$$dv(S) = \prod_i ds_{ii} \prod_{i<j} ds_{ij}$$

and c_N is the normalizing constant chosen in such a way that

$$\int_{\mathcal{S}_N} d\mu(S) = 1.$$

5. Find the value of the constant c_N.

Exercise 3.2.12 (Continuation of Exercise 3.2.9). 1. Compute the integrals

$$\int_{\mathcal{S}_N} \mathrm{tr}(S^2) d\mu(S), \qquad \int_{\mathcal{S}_N} \mathrm{tr}(S^4) d\mu(S).$$

2. Draw all (both orientable and non-orientable) surfaces glued from the 2-gon and from the square.

3. Find the number of all (orientable and non-orientable) gluings of the $2n$-gon.

4. Prove that

$$\sum_\sigma N^{V(\sigma)} = 2^n \int_{\mathcal{S}_N} \mathrm{tr}(S^{2n}) d\mu(S),$$

where the sum on the left-hand side is taken over all possible (orientable and non-orientable) gluings of the $2n$-star.

(2) Give a combinatorial class generalizing one-face maps which corresponds to not-necessarily orientable polygon gluings (as in the previous question), in the way that one-face maps correspond to orientable polygon gluings.