Midterm

Solutions

Instructions:

- All answers must be justified, unless otherwise stated.
- Unless otherwise stated, you may use any result proved or stated in class but you should be explicit about which result you are using.
- For full marks you should answer **question 1** and **any three of questions 2,3,4 and 5**. If you answer more than three of 2,3,4 and 5 only the first three will be graded.
- No collaboration is allowed.
- Please put your solutions in the space provided. If you need more space use the last page and clearly indicate the problem number your solution corresponds to.

1. On this question, no explanation is required for full marks, however, if you provide explanation you may be able to obtain partial marks in the case of an incorrect answer.

(a) (2 points) Let $A(x)$ and $B(x)$ be formal power series. Which of the following are valid formal power series operations under the indicated restrictions

(i) $A(1+B(x))$ with $B(0) = 0$ (ii) $A(1)$ with A a polynomial (iii) $A'(x)$ (iv) $B(x)+B(x)^2+B(x)^3+\cdots$

Invalid (unless A is a polynomial) valid Valid invalid (unless $B(0)=0$)

(b) (2 points) Which of the following are specifications for the class of Dyck paths?

(i) $\mathcal{D} = \textsc{Seq}(\nearrow \times \mathcal{D} \times \searrow)$ (ii) $\mathcal{D} = \mathcal{E} \cup (\nearrow \times \textsc{Seq}(\mathcal{D}) \times \searrow)$ (iii) $\mathcal{D} = \mathcal{E} \cup (\nearrow \times \mathcal{D} \times \searrow \times \mathcal{D})$ (iv) $\mathcal{D} = \nearrow \times \mathcal{D} \times \searrow \times \mathcal{D}$

yes no yes no

(c) (2 points) Let $\mathcal{C}$ be a combinatorial class and $(\omega_1(c), \omega_2(c))$ a weight function on $\mathcal{C}$. Give a formula (using the bivariate generating function) for the average value of ω_2 among all elements of $\mathcal{C}$ with $\omega_1(c) = n$.

Let $C(x,y)$ be the bivariate generating function

the average is $\dfrac{[x^n] \frac{\partial}{\partial y} C(x,y)\big|_{y=1}}{[x^n]\, C(x,1)}$

(d) (2 points) Give a classical binomial identity to which the following is a q-analogue:

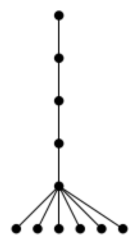
$$\begin{bmatrix} a+1+b \\ b \end{bmatrix}_q = \sum_{j=0}^{b} q^{(a+1)(b-j)} \begin{bmatrix} a+j \\ j \end{bmatrix}_q$$

$$\binom{a+1+b}{b} = \sum_{j=0}^{b} \binom{a+j}{j} .$$

(e) (2 points) Give the generating function for partitions where every part is even and there are no more than two copies of any part.

$$\prod_{j=0}^{\infty} (1 + x^{2j} + x^{4j})$$

2. *(Answer any three of questions 2,3,4 and 5.)* Let $\mathcal{B}$ be the combinatorial class of rooted trees where exactly one vertex has more than 1 child, and all the children of this vertex are leaves. Such trees look like brooms.



(a) (2 points) Give a specification for $\mathcal{B}$.

$$\mathcal{B} = \underbrace{(\mathcal{Z} \times Seq(\mathcal{Z}))} \times \underbrace{(\mathcal{Z} \times \mathcal{Z} \times Seq(\mathcal{Z}))}$$

this represents the handle which must have at least one vertex (namely the vertex with more than one child)

This represents the bristles. There must be at least two of them so that the vertex with more than one child in fact has more than one child

(b) (1 point) Is $\mathcal{B}$ regular? Explain.

Yes the definition is iterative and uses only Seq, $\mathcal{Z}$, $\times$ so $\mathcal{B}$ is regular

(c) (2 points) Give a class of binary strings which has a size-preserving bijection to $\mathcal{B}$. Explain your answer, but you do not need to prove the bijection.

Take the class of binary strings with at least two 1s and at least one 0 and where all 0s come before all 1s

The bijection is $0^n 1^m \longmapsto$

3. (5 points) *(Answer any three of questions 2,3,4 and 5.)*

Let $\exp(x)$ be the formal power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Prove that

$$\exp(x)^{-1} = \exp(-x)$$

as formal power series.

It suffices to prove that

$\exp(x)\exp(-x) = 1$ by definition of multiplicative inverse.

So calculate

$$\begin{aligned}
&\exp(x)\exp(-x) \\
&= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\frac{(-1)^{n-k}}{(n-k)!}\right) x^n && \text{by definition of multiplication for formal power series} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}\right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}\right) \frac{x^n}{n!} && \text{by the binomial theorem applied to } (1-1)^n \\
&= 1
\end{aligned}$$

the result follows

4. (5 points) *(Answer any three of questions 2,3,4 and 5.)*

Fix $c \in \mathbb{Z}_{>0}$. For each $n \in \mathbb{Z}_{\geq 0}$ find the number of ordered rooted trees where each vertex has a number of children divisible by c. Binomial coefficients in your answer should not have negative or fractional arguments.

This class of trees has specification

$$\mathcal{T} = \mathcal{Z} \times (\mathcal{E} \cup \mathcal{T}^{c} \cup \mathcal{T}^{2c} \cup \mathcal{T}^{3c} \cup \dots)$$
$$= \mathcal{Z} \times \mathrm{Seq}(\mathcal{T}^{c})$$

so $T(x) = x \dfrac{1}{1 - T(x)^c}$

Now apply LIFT

$$[x^n]T(x) = \frac{1}{n}[u^{n-1}]\left(\frac{1}{1-u^c}\right)^n$$
$$= \frac{1}{n}[u^{n-1}](1-u^c)^{-n}$$
$$= \frac{1}{n}\begin{cases} \binom{-n}{\frac{n-1}{c}} & \text{if } c \mid (n-1) \\ 0 & \text{otherwise} \end{cases}$$

note these are not fractional arguments because $c \mid (n-1)$

$$= \frac{1}{n}\begin{cases} \binom{n + \frac{n-1}{c} - 1}{\frac{n-1}{c}} & \text{if } c \mid (n-1) \\ 0 & \text{otherwise} \end{cases}$$

because $\binom{-n}{k} = \binom{n+k-1}{k}$

5. (5 points) *(Answer any three of questions 2,3,4 and 5.)* Give a combinatorial proof that

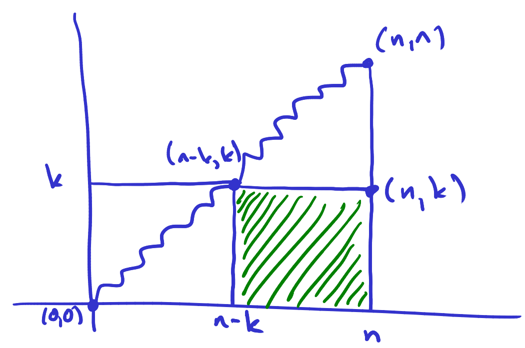
$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{L}(a,b)$ be the set of lattice paths from $(0,0)$ to (a,b) using steps $\uparrow$, $\rightarrow$

we have $\mathcal{L}(n,n) \cong \bigcup_{k=0}^{n} \mathcal{L}(n-k,k) \times \mathcal{L}(k,n-k)$

by taking a lattice path in $\mathcal{L}(n,n)$ and decomposing it into the first n steps (a lattice path from $(0,0)$ to $(n-k,k)$ for some k) and the next n steps shifted to begin at $(0,0)$ and hence becoming a lattice path to $(k,n-k)$



The inverse map is to concatenate the two paths. and any $0 \leq k \leq n$ is possible

Furthermore the area of the block marked in green is $(n-(n-k))k = k^2$

Thus if $P \mapsto (P_1, P_2)$ is an instance of this decomposition then

$$\text{area}(P) = \text{area}(P_1) + \text{area}(P_2) + k^2$$

so
$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{P \in \mathcal{L}(n,n)} q^{\text{area}(P)} = \sum_{k=0}^{n} \Big(\sum_{P_1 \in \mathcal{L}(n-k,k)} q^{\text{area} P_1} \Big)\Big(\sum_{P_2 \in \mathcal{L}(k,n-k)} q^{\text{area} P_2} \Big) q^{k^2}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{k^2}$$

which is what we wanted to prove.

Extra space for solutions.