CO 330, LECTURE 5 SUMMARY

FALL 2017

Summary

To go with the sum lemma and the product lemma, we defined notation for two special small classes.

Definition 1.

- *Let $\mathcal{E}$ be a combinatorial class consisting of a single element (usually written ϵ) where that single element has size* 0.
- *Let $\mathcal{Z}$ be a combinatorial class consisting of a single element where that single element has size* 1.

This notation is useful and lets us rewrite our specification of our class of binary rooted trees as

$$\mathcal{B} \simeq \mathcal{Z} \times (\mathcal{B} \cup \mathcal{E})^2$$

which we can immediately translate into

$$B(x) = x(B(x) + 1)^2$$

using the sum rule, product rule and the generating functions for $\mathcal{E}$ and $\mathcal{Z}$.

Next we set things up to define the sequence rule

Definition 2. *Let $\mathcal{A}$ be a combinatorial class, we write $\mathcal{A}^k$ for $\underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{k \text{ times}}$ and let $\mathcal{A}^0 = \mathcal{E}$.*

Definition 3. *Let $\mathcal{A}$ be a combinatorial class with $\mathcal{A}_0 = \emptyset$, then $A^* = Seq(\mathcal{A}) = \bigcup_{\ell=0}^{\infty} \mathcal{A}^\ell$*

The point is that $\mathrm{Seq}(\mathcal{A})$ is a combinatorial class with size function the sum of the sizes of the elements in a sequence and that the generating function of $\mathrm{Seq}(\mathcal{A})$ is $\frac{1}{1-A(x)}$. For the rigorous foundations of the last part see your assignment.

Next we looked at two examples. First we had ordered rooted trees, that is rooted trees where there is an order on the children at each vertex and size is the number of vertices. This class has the specification

$$\mathcal{T} \simeq \mathcal{Z} \times \mathrm{Seq}(\mathcal{T})$$

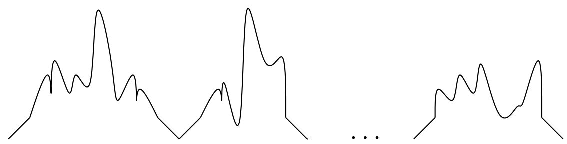
and so have generating function satisfying

$$T(x) = \frac{x}{1 - T(x)}$$

rearranging and solving the resulting quadratic equation we get

$$[x^n]T(x) = \frac{1}{n}\binom{2n-2}{n-1}$$

The second example was Dyck paths which are paths starting at $(0,0)$, using the steps $\nearrow$ and $\searrow$ which end on the x-axis and never go below the x-axis. We can decompose these by breaking them each time they return to the origin, like this



giving the specification

$$\mathcal{D} \simeq \mathrm{Seq}(\mathcal{Z}_{\nearrow} \times \mathcal{D} \times \mathcal{Z}_{\searrow})$$

and so

$$D(x) = \frac{1}{1 - x^2 D(x)}.$$

Rearranging and solving the resulting quadratic equation we get

$$[x^{2n}]D(x) = \frac{1}{n}\binom{2n-2}{n-1}$$

and $[x^{2n+1}]D(x) = 0$.

Taking all the examples mentioned today together we get

$$|\mathcal{B}_n| = |\mathcal{T}_{n+1}| = |\mathcal{D}_{2n}|$$

so we must have bijections between these combinatorial classes (which shift the weights appropriately).

References

The course notes do not use $\mathcal{Z}$ and $\mathcal{E}$, that is more of a Flajolet and Sedgewick notation (see their book "Analytic Combinatorics). The course notes do the sequence rule (using the Kleene star notation) in Lemma 4.12 and Proposition 4.13, and the classes counted by Catalan numbers are in chapter 6. Instead of Dyck paths, the course notes rotate everything and end up with super diagonal lattice walks (SDLWs), and the ordered rooted trees are called plane planted trees (PPTs).

Have you noticed that in the course notes, after the exercises, in some sections there are some end notes which provide some interesting comments and references?