CO 330, LECTURE 4 SUMMARY

FALL 2017

Summary

Today we defined what it means for a operation on formal power series to be valid:

Definition 1. *Suppose* $\Theta : R[[x]]^k \to R[[x]]$. Θ *is* valid *if for all* $n \in \mathbb{Z}_{\geq 0}$ *there exists an* $m \in \mathbb{Z}_{\geq 0}$ *such that for any input series* $f_1(x) = \sum_{i\geq 0} a_i^{(1)} x^i$, $f_2(x) = \sum_{i\geq 0} a_i^{(2)} x^i$, $\dots$, $f_k(x) = \sum_{i\geq 0} a_i^{(k)} x^i$, *the coefficient*

$$[x^n]\Theta(f_1, f_2, \dots, f_k)$$

depends only on $a_0^{(1)}, a_1^{(1)}, \dots, a_m^{(1)}, a_0^{(2)}, a_1^{(2)}, \dots, a_m^{(2)}, \dots, a_0^{(k)}, a_1^{(k)}, \dots, a_m^{(k)}$.

Intuitively the point is that if you want to calculate some particular coefficient of the answer then you only need a finite amount of information from the input series.

We noticed that addition and multiplication are valid operations. So is evaluation at 0, but evaluation at 1 is not valid. You did some more examples in groups, the most important of which is probably composition which the middle groups did. They noticed that composition is not valid. However, if you restrict the input series then it is valid on this smaller domain. Either the outer series needs to be a polynomial or the inner series needs to have zero constant term in order for composition to be valid.

Note that an operation could still make sense analytically even if it is not a valid formal power series operation (like evaluating at 1). These things can still be useful in enumerative combinatorics, but you need to step outside the purely formal framework to use them (this is known as analytic combinatorics).

We finished off the class by writing and proving the sum lemma and product lemma in our notation. The result was that given combinatorial classes $\mathcal{B}$ and $\mathcal{C}$ we could define classes $\mathcal{B} + \mathcal{C}$ (when $\mathcal{B} \cap \mathcal{C} = \emptyset$) and $\mathcal{B} \times \mathcal{C}$ and the generating functions for these classes were $B(x) + C(x)$ and $B(x)C(x)$ respectively.

References

The course notes do not talk about valid formal power series operations. Instead they talk about *convergence of a sequence of formal series* (see definition 7.10). The point of either the notion of valid or this notion of convergence of a sequence of formal series is to rigorously define when notions like composition of series makes sense. The way the course notes does it is more powerful but also, I think, harder to understand, while the simpler notion of valid operation is sufficient for what we'll need.

For the sum lemma and the product lemma see Propositions 4.8 and 4.9 of the notes.