CO 330, LECTURE 3 SUMMARY

FALL 2017

Summary

Today we set up formal power series. We defined ring, ring of polynomials, ring of formal power series, and ring of formal Laurent series. The latter three definitions are almost the same, for the first the expressions start at 0 and are of finite length, for the second they start at 0 and are infinite, for the last they can start at any integer. Here they are:

Definition 1. *Let R be a ring. The ring of polynomials over R, denoted $R[x]$ is the set of expressions of the form*

$$a_0 + a_1x + \cdots a_nx^n$$

with $a_0, a_1, \ldots, a_n \in R$ and $n \in \mathbb{Z}_{\geq 0}$ with operations

$$(a_0+a_1x+\cdots a_nx^n)+(b_0+b_1x+\cdots b_mx^m) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_{\max(m,n)}+b_{\max(m,n)})x^{\max(m,n)}$$

where $a_\ell = 0$ for $\ell > n$ and $b_\ell = 0$ for $\ell > m$, and

$$(a_0 + a_1x + \cdots a_nx^n)(b_0 + b_1x + \cdots b_mx^m) = \sum_{k=0}^{m+n} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) x^k$$

Definition 2. *Let R be a ring. The ring of formal power series over R, denoted $R[[x]]$ is the set of expressions of the form*

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

with $a_0, a_1, a_2, \ldots \in R$ and with operations

$$(\sum_{n=0}^{\infty} a_nx^n) + (\sum_{m=0}^{\infty} b_mx^m) = \sum_{k=0}^{\infty}(a_k + b_k)x^k$$

and

$$(\sum_{n=0}^{\infty} a_nx^n)(\sum_{m=0}^{\infty} b_mx^m) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell} \right) x^k$$

Definition 3. *Let R be a ring. The ring of formal Laurent series over R, denoted $R((x))$ is the set of expressions of the form*

$$\sum_{n=I}^{\infty} a_nx^n$$

with $a_0, a_1, a_2, \ldots \in R$ and $I \in \mathbb{Z}$, with operations

$$(\sum_{n=I}^{\infty} a_nx^n) + (\sum_{m=J}^{\infty} b_mx^m) = \sum_{k=\min(I,J)}^{\infty} (a_k + b_k)x^k$$

where $a_\ell = 0$ *for* $\ell < I$ *and* $b_\ell = 0$ *for* $\ell < J$*, and*

$$(\sum_{n=I}^{\infty} a_n x^n)(\sum_{m=J}^{\infty} b_m x^m) = \sum_{k=I+J}^{\infty} \left(\sum_{\ell=I}^{k-J} a_\ell b_{k-\ell} \right) x^k$$

You can iterate these constructions and we can use that to define generalized binomial expansions as follows:

Definition 4. *In* $\mathbb{Q}[y]$ *define*

$$\binom{y}{n} = \frac{y(y-1)\cdots(y-(n-2))(y-(n-1))}{n!}$$

for $n \in \mathbb{Z}_{\geq 0}$.

Definition 5. *In* $(\mathbb{Q}[y])[[x]]$ *define*

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n$$

Throughout, these definitions make sense in and of themselves at a purely formal level, but they are also good definitions in a more human sense because they line up with the calculus functions that are written using the same or similar notation when those calculus functions are defined. That is why we use this kind of notation for formal power series instead of just writing them as sequences of coefficients and defining the ring operations directly on these sequences.

References

The beginning of chapter 7 of the course notes covers this material.