CO 330, LECTURE 29 SUMMARY

FALL 2017

Summary

Today we began Boltzmann samplers. The point of Boltzmann samplers is that you give up getting an exact size for your output in exchange for a faster algorithm. Also it uses the generating function and the specification in a neat way. First we need to set up some theory.

For this section we will need to think of the generating function not just as a formal power series but also as series in the sense of calculus. We'll need to know the radius of convergence and plug in values. Also, every class is unlabelled for the next few lectures. We'll come back to how the labelled situation differs.

Definition 1. *A* discrete probability space *with sample space $\mathcal{S}$ and probability function P is a countable set $\mathcal{S}$ and a function P from the set of subsets of $\mathcal{S}$ to $[0, 1]$ satisfying*

- $P(\mathcal{S}) = 1$
- *If $A = \{a_1, a_2, \ldots\}$ then $P(A) = \sum_{i=1}^{\infty} P(a_i)$.*

Note that in order to define P it suffices to specify $P(s)$ for all $s \in \mathcal{S}$.

We want a particular discrete probability space where each element of a combinatorial class is weighted by its contribution to the generating function.

Definition 2. *Let $\mathcal{C} \neq \emptyset$ be a combinatorial class and let $\rho > 0$ be the radius of convergence of $C(x)$. The* Boltzmann model *at x for $0 < x < \rho$ is a discrete probability space with sample space $\mathcal{C}$ and probability function given by*

$$P_x(c) = \frac{x^{|c|}}{C(x)} \quad \text{for } c \in \mathcal{C}$$

We checked in class that this really does give a discrete probability space (this just uses the definitions above along with the definition of generating function).

Probabilities are uniform by size in the Boltzmann model:

Proposition 3. *Let $\mathcal{C} \neq \emptyset$ be a combinatorial class and $0 < x < \rho$ where ρ is the radius of convergence of $C(x)$. Let P_x be the probability function of the Boltzmann model. Suppose $c, c' \in \mathcal{C}$ with $|c| = |c'|$ then $P_x(c) = P_x(c')$.*

The proof is direct from the definition. We can also calculate the expected size (again a direct calculation from the definitions).

Proposition 4. *Let $\mathcal{C}$, ρ, x, P_x be as above. The expected size is*

$$E_x = x\frac{C'(x)}{C(x)}$$

Now we will suppose our combinatorial class has

$$\lim_{x\to\rho^-} x\frac{C'(x)}{C(x)} = \infty$$

because then for any $n > 0$ there will be some x so that the Boltzmann model at x will have expected size n.

The next step is to actually build the Boltzmann sampler. Similarly to recurive generation we will build it recursively from the specification. We only got a start on this in today's lecure.

Consider the class $\mathcal{E}$, then no matter what x is ϵ is generated with probability $\frac{x^0}{E(x)} = \frac{1}{1} = 1$. So

```
def BoltzmannE
  input x
  return E
```

Similarly for $\mathcal{Z}$ the single atom is generated with probability $\frac{x^1}{Z(x)} = \frac{x}{x} = 1$. So

```
def BoltzmannZ
  input x
  return Z
```

Now suppose $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ with $\mathcal{B} \cap \mathcal{C} = \emptyset$. What is the probability that $a \in \mathcal{A}$ came from $\mathcal{B}$? Just sum the possibilities: $P_x(\mathcal{B}) = \sum_{b\in\mathcal{B}} x^{|b|}/A(x) = B(x)/A(x)$. So

```
def BoltzmannA=BcupC
  input x
  u = rand()
  if u < B(x)/A(x)
    return Boltzmann B
  else
    return Boltzmann C
```

So far these are not so different from recursive generation; we'll see a bigger improvement with $\times$. Also note we do need to be able to evaluate the generating functions at x. More on this later.

References

Today we began http://people.math.sfu.ca/~kyeats/teaching/math343/12-343.pdf