CO 330, LECTURE 27 SUMMARY

FALL 2017

Summary

We started by looking at labelled trees and labelled rooted trees and calculated that the number of labelled (unrooted) trees on n vertices is n^{n-2} which is a classic formula that we will return to bijectively in a couple of weeks.

Then we did some problems in groups. The problems, solutions, and references are included below (with the error that the green one had in class corrected).

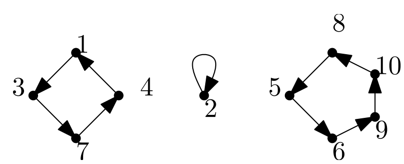
r-cycle permutations. (The was the one printed on purple paper.)

Let $\mathcal{P}^{(r)}$ be the class of permutations that decompose into r cycles

- Give an example of an element of $\mathcal{P}^{(r)}$ of size 10.
- Give a specification for $\mathcal{P}^{(r)}$ using the operations we discussed in class.
- Write the exponential generating function for $\mathcal{P}^{(r)}$

Permutation answers

- Many answers are possible, here's one



- $\mathcal{P}^{(r)} = \textsc{Set}_r(\textsc{Cyc}(\mathcal{Z}))$
- $P^{(r)}(x) = \frac{1}{r!}\left(\log\left(\frac{1}{1-x}\right)\right)^r$.

(Reference: example II.12 on p121 of Flajolet and Sedgewick "Analytic Combinatorics")

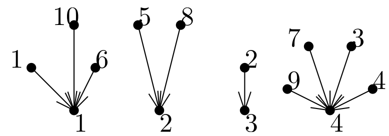
Surjections. (This was the one printed on blue paper.)

Let $\mathcal{R}_n^{(r)}$ be the class of surjections (that is, onto maps) from $\{1,\dots,n\}$ onto $\{1,\dots,r\}$ and let $\mathcal{R}^{(r)} = \bigcup_{n\geq 0}\mathcal{R}_n^{(r)}$ as a combinatorial class with n as the size.

- Draw an element of $\mathcal{R}_{10}^{(4)}$.
- Thinking of these as labelled objects with the elements of $\{1,\dots,n\}$ as the atoms, give a specification for $\mathcal{R}^{(r)}$ using the operations we discussed in class. *Hint, think about the preimages $f^{-1}(1),\dots,f^{-1}(r)$.*
- Write down the exponential generating function for $\mathcal{R}^{(r)}$ and for $\mathcal{R} = \bigcup_{r\geq 0}\mathcal{R}^{(r)}$.

Surjection answers

- Many answers are possible, here's one



- $\mathcal{R}^{(r)} = \textsc{Seq}_r(\textsc{Set}_{\geq 1}(\mathcal{Z}))$
- $R^{(r)}(x) = (e^x - 1)^r$, $R(x) = \frac{1}{2-e^x}$.

(Reference: Flajolet and Sedgewick "Analytic Combinatorics" p107)

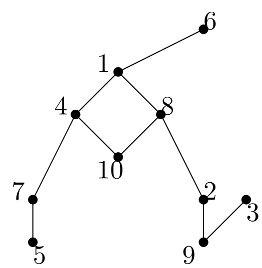
1-cycle trivalent graphs. (This was the one printed on green paper.)

Let $\mathcal{Q}$ be the class of connected labelled graphs with exactly one cycle and maximum vertex degree 3 and where all vertices of degree 3 are on the cycle.

- Draw an example of an element of Q of size 10.
- Give a specification for $\mathcal{Q}$ using the operations we discussed in class. You may use $\frac{1}{2}\textsc{Cyc}$ for undirected cycles (since there are exactly two directed cycles for each undirected cycle).
- Write the exponential generating function for $\mathcal{Q}$
- Calculate the number of these graphs of size n.

1-cycle answers

- Many answers are possible, here's one



- $\mathcal{Q} = \frac{1}{2}\textsc{Cyc}_{\geq 3}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$
- $Q(x) = \frac{1}{2}\left(\log\left(\frac{1}{1-\frac{x}{1-x}}\right) - \frac{x}{1-x} - \frac{x^2}{2(1-x)^2}\right)$
- $(n-1)!\left(2^n - 1 - \binom{n+1}{2}\right)$ for $n \geq 2$ and 0 otherwise

(Reference: example 11.22 in the course notes.)

References

The tree example we started with is example 11.18 in the course notes. The three group problems have their references listed above.