CO 330, LECTURE 26 SUMMARY

FALL 2017

Summary

We started by revisiting old examples in the language from last class. Specifically. If we think of permutations in one-line notation we get the specification

$$\mathcal{P} = \textsc{Seq}(\mathcal{Z})$$

and so $P(x) = \frac{1}{1-x}$. Cycles we can build similarly

$$\mathcal{C} = \textsc{Cyc}(\mathcal{Z})$$

and so $C(x) = \log(\frac{1}{1-x})$. We can also think of permutations in disjoint cycle form which gives the decomposition

$$\mathcal{P} = \textsc{Set}(\mathcal{C})$$

and so $P(x) = \exp(\log(\frac{1}{1-x})) = \frac{1}{1-x}$.

As I commented before this is a very general phenomenon: if $\mathcal{A}$ is connected whatevers (say connected graphs or trees) then $\textsc{Set}(\mathcal{A})$ is potentially disconnected whatevers (say graphs or forests), so exp always takes us from connected to not necessarily connected.

Next we defined one last labelled operation

Definition 1. *Let $\mathcal{C}$ be a labelled combinatorial class. Then define the* ***pointed*** *or* ***rooted*** *version of $\mathcal{C}$, denoted $\mathcal{C}^\bullet$ to be the set of pairs (c, z) where $c \in \mathcal{C}$ and z is an atom of c. The size of (c, z) is the size of c.*

What this does it pick out a distinguished atom, like the root of a rooted tree. We will often draw this by putting an arrow pointing at the distinguished atom.

The most classic example of this is to get rooted trees from trees. We worked out what this means in the example of trees with three vertices in class.

Proposition 2. *Let $\mathcal{C}$ be a labelled combinatorial class. Then*

$$C^\bullet(x) = x\frac{d}{dx}C(x)$$

Proof. For $c \in \mathcal{C}$, $|c|$ is the number of atoms in c. Therefore

$$\begin{aligned} C^\bullet(x) &= \sum_{\substack{(c,z) \\ c\in\mathcal{C} \\ z \text{ an atom of } c}} x^{|c|} \\ &= \sum_{c\in\mathcal{C}} |c| x^{|c|} \\ &= x\frac{d}{dx}C(x) \end{aligned}$$

□

Next we worked out what is going on with set partitions. Given a set S, a set partition of S is a decomposition of S into nonempty disjoint subsets whose union is S. There is no order structure on the subsets. For example the set partitions of $\{1, 2, 3\}$ are:

$$\{1,2,3\} \qquad \{1,2\},\{3\} \qquad \{1,3\},\{2\} \qquad \{1\},\{2,3\} \qquad \{1\},\{2\},\{3\}$$

Definition 3. *The number of set partitions of $\{1, \dots, n\}$ into k parts is denoted*

$$\left\{ {n \atop k} \right\}$$

*and these numbers are called the **Stirling partition numbers** or Stirling numbers of the second kind.*

We can specify the class of set partitions using the operations from last time. First of all the parts are nonempty subsets of atoms:

$$\textsc{Set}_{\geq 1}(\mathcal{Z})$$

Let $\mathcal{S}_k$ be set partitions with k parts (where the size is the n of $\{1, \dots, n\}$, the set being partitioned), then

$$\mathcal{S}_k = \textsc{Set}_k(\textsc{Set}_{\geq 1}(\mathcal{Z}))$$

so

$$\sum_{n \geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = S_k(x) = \frac{(e^x - 1)^k}{k!}$$

Also let $\mathcal{S}$ be arbitrary set partitions, then

$$\mathcal{S} = \textsc{Set}(\textsc{Set}_{\geq 1}(\mathcal{Z})$$

so

$$S(x) = e^{(}e^x - 1)$$

The coefficients of $S(x)$ are called Bell numbers.

Finally, we ran out of time in class to get a formula for the Stirling partition numbers, but you can do so from the above formula for $S_k(x)$. The calculation goes like this:

$$\frac{1}{n!} \left\{ {n \atop k} \right\} = \frac{1}{k!} [x^n] \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{(k-j)x} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(k-j)^n}{n!}$$

so

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

References

We are continuing in the course notes chapter 11 or `http://people.math.sfu.ca/~kyeats/teaching/math343/labelled.pdf`.